Problem Set 1
EE221a: Linear Systems Theory

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1. Give two examples of infinite dimensional linear spaces. In each case, show that they are infinite dimensional by exhibiting an infinite family of linearly independent vectors.

2. Given $A, B, C, X \in \mathbb{C}^{n\times n}$. Consider the following maps from $\mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ and determine if they are linear or not. Give a brief proof if true and a counterexample if false.

   (a) $X \mapsto AX + XB$

   (b) $X \mapsto AX + BXC$

   (c) $X \mapsto AXA - X$

   (d) $X \mapsto AX + XBX$

3. **Convolution Maps**.
   Given a function $h : \mathbb{R}^2 \mapsto \mathbb{R}$ with $h(t, \tau) = 0$ for $t \leq \tau$ and

   $$|h(t,\tau)| \leq M \exp^{-\alpha(t-\tau)}$$

   for some $M, \alpha > 0$, consider the linear map $\mathcal{L}$ defined by

   $$(\mathcal{L}u)(t) := \int_{-\infty}^{t} h(t,\tau)u(\tau)d\tau$$

   Show that

   (a) $\mathcal{L} : L_\infty(-\infty,\infty) \mapsto L_\infty(-\infty,\infty)$. That is, prove that if $u \in L_\infty(-\infty,\infty)$ then $y \in L_\infty(-\infty,\infty)$.

   (b) $\mathcal{L}$ is a linear map.

4. **Sylvester Inequality**
   Given $A, B \in \mathbb{R}^{n\times n}$, show that

   $$\text{rank } B - \dim\left[\mathcal{N}(B)\right] \leq \text{rank } AB \leq \min\left[\text{rank } A, \text{rank } B\right]$$

5. Let $\mathcal{R}$ be the set of *proper* (i.e. bounded at $s = \infty$), *stable* (i.e. analytic (no poles) in the closed right half complex plane, $\mathbb{C}_+$) rational functions. Prove that $\mathcal{R}$ is a commutative ring? Is it a field: if true, prove; if false, give a counterexample. To generalize, let $U \subset \mathbb{C}$ be an arbitrary region in the complex plane, and let $\mathcal{R}_U$ be the set of proper rational functions which are analytic in U. Is $\mathcal{R}_U$ a ring? Is it a field?

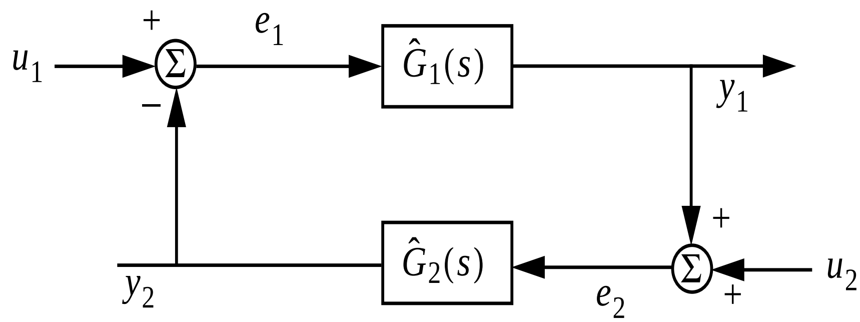

Figure 1: A feedback system with inputs u_1, u_2, outputs y_1, y_2 and error signals e_1, e_2.

6. **Feedback Systems**
Let $G_1(s) \in \mathcal{R}_U^{n_o \times n_i}, G_2(s) \in \mathcal{R}_U^{n_i \times n_o}$. Further, assume that G_1 is *strictly proper*, that is, it $\to 0$ as $s \to \infty$. Consider the feedback system shown in the figure. Let $u_i(s), e_i(s), y_i(s)$ stand for the Laplace transforms of the various signals in the loop, for $i = 1, 2$. Show that

   (a) The transfer function $H_{yu} : (u_1(s), u_2(s)) \mapsto (y_1(s), y_2(s))$ is given by

$$\begin{bmatrix} G_2(I + G_1G_2)^{-1} & -G_2G_1(I + G_2G_1)^{-1} \\ G_1G_2(I + G_1G_2)^{-1} & G_1(I + G_2G_1)^{-1} \end{bmatrix}$$

   (b) $H_yu \in \mathcal{R}_U^{(n_o+n_i)\times(n_o+n_i)} \Leftrightarrow \det\,(I + G_1G_2)(s) \neq 0$ for all $s \in U$.