1. Satellite Problem

Model the earth and satellite as particles. The normalized equations of motion simplified to 2 dimensions (from Lagrange’s equations of motion, the Lagrangian $L = T − V = \frac{1}{2}r^2 + \frac{1}{2}r^2\dot{\theta}^2 − \frac{k}{r}$):

\begin{align*}
\ddot{r} &= r\dot{\theta}^2 − \frac{k}{r^2} + u_1 \\
\ddot{\theta} &= −2\dot{\theta}r + \frac{1}{r}u_2
\end{align*}

with $u_1, u_2$ representing the radial and tangential forces due to thrusters. The reference orbit with $u_1 = u_2 = 0$ is circular with $r(t) \equiv p$ and $\theta(t) = \omega t$. From the first equation it follows that $p^3\omega^2 = k$.

Obtain the linearized equation about this orbit. (How many state variables are there?)

2. Lie Brackets

Define the Lie Bracket of two matrices $A, B \in \mathbb{R}^{n \times n}$ to be a new matrix $[A, B] := AB − BA$. Show that for any 3 matrices $A, B, C \in \mathbb{R}^{n \times n}$ we have

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

This is called the Jacobi Identity.

3. Peano Baker formula

For a linear time varying system, prove that

$$\Phi(t, t_0) = I + \int_{t_0}^{t} A(\sigma_1) d\sigma_1 + \int_{t_0}^{t} \int_{\sigma_1}^{t} A(\sigma_2) d\sigma_2 d\sigma_1 + \cdots$$

4. From Brockett[70]

Matrix differential equations are natural tools for some physical problems. For example, consider the problem of describing the orientation of one set of coordinate axes (labeled $x_1, x_2, x_3$) with respect to a second set of axes (labeled $y_1, y_2, y_3$). Say that the projection of the $y_j$ axis of a unit vector along the $x_i$ axis is $r_{ij}$. There are 9 such direction cosines and we arrange them in a matrix $R \in \mathbb{R}^{3 \times 3}$. Consider, the $x$ system to be fixed or the ground axes. If the $y$ system is rotating about the $x$ axes, with angular velocity $\omega_i$ about axis $x_i$ for for $i = 1, 2, 3$, respectively, then we have

$$\dot{R}(t) = \begin{bmatrix}
0 & \omega_3 & -\omega_2 \\
-\omega_3 & 0 & \omega_1 \\
\omega_2 & -\omega_1 & 0
\end{bmatrix} R(t)$$

Find the state transition matrix for $\omega_3 = c_3, \omega_2 = c_2 \sin \omega t, \omega_1 = c_1 \cos \omega t$. This corresponds to a rigid body in free spinning motion. (Ask your TA for a hint.)
5. **From Brockett[70]**

Suppose that the boundary conditions for $\dot{x} = A(t)x(t)$ were specified in part at $t_0$ and in part at $t_1$. In particular suppose that

$$Mx(t_0) + Nx(t_1) = b$$

with $\text{Rank } (M, N) = n$ (dimension of $x$). Show that this two point boundary value problem has a unique solution if $M + N\Phi(t_1, t_0)$ is nonsingular.

6. **H dot or Lax Equation** Let $\Omega \in \mathbb{R}^{n \times n}$ be a skew symmetric matrix and $H(0) \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Show that the differential equation

$$\dot{H} = [H, \Omega]$$

generates a flow on symmetric matrices, that is $H(t)$ is symmetric for all $t$. Also show that the eigenvalues of $H(t)$ are the eigenvalues of $H(0)$. These flows are called *iso-spectral.*