Problem Set 3
EE221a: Linear Systems Theory

Prof. S.S. Sastry

Fall 2005

Issued: September 22nd, 2005. **Due: October 4th, 2005**.

1. **Satellite Problem**

   Model the earth and satellite as particles. The *normalized* equations of motion simplified to 2 dimensions (from Lagrange's equations of motion, the Lagrangian $L = T - V = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - \frac{k}{r}$):

$$\begin{aligned} \ddot{r} &= \; r\dot{\theta}^2 - \frac{k}{r^2} + u_1 \\ \ddot{\theta} &= \; -2\frac{\dot{\theta}}{r} + \frac{1}{r}u_2 \end{aligned}$$

   with u_1, u_2 representing the radial and tangential forces due to thrusters. The reference orbit with $u_1 = u_2 = 0$ is circular with $r(t) \equiv p$ and $\theta(t) = \omega t$. From the first equation it follows that $p^3\omega^2 = k$. Obtain the linearized equation about this orbit. (How many state variables are there ?)

2. **Lie Brackets**

   Define the Lie Bracket of two matrices $A, B \in \mathbb{R}^{n\times n}$ to be a new matrix $[A, B] := AB - BA$. Show that for any 3 matrices $A, B, C \in \mathbb{R}^{n\times n}$ we have

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

   This is called the Jacobi Identity.

3. **Peano Baker formula**

   For a linear time varying system, prove that

$$\begin{aligned} \Phi(t,t_0) = \; & I + \int_t^{t_0} A(\sigma_1)d\sigma_1 + \int_{t_0}^t A(\sigma_1)\left[\int_{t_0}^{\sigma_1} A(\sigma_2)d\sigma_2\right]d\sigma_1 + \\ & \int_{t_0}^t A(\sigma_1)\left[\int_{t_0}^{\sigma_1} A(\sigma_2)\left[\int_{t_0}^{\sigma_2} A(\sigma_3)d\sigma_3\right]d\sigma_2\right]d\sigma_1 + \cdots \end{aligned}$$

4. **From Brockett[70]**

   Matrix differential equations are natural tools for some physical problems. For example, consider the problem of describing the orientation of one set of coordinate axes (labeled x_1, x_2, x_3) with respect to a second set of axes (labeled y_1, y_2, y_3). Say that the projection of the y_j axis of a unit vector along the x_i axis is r_{ij}. There are 9 such direction cosines and we arrange them in a matrix $R \in \mathbb{R}^{3\times 3}$. Consider, the x system to be fixed or the ground axes. If the y system is rotating about the x axes, with angular velocity ω_i about axis x_i for for $i = 1, 2, 3$, respectively, then we have

$$\dot{R}(t) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} R(t)$$

   Find the state transition matrix for $\omega_3 \equiv c_3$, $\omega_2 = c_2 \sin\omega t$, $\omega_1 = c_1 \cos\omega t$. This corresponds to a rigid body in free spinning motion. (Ask your TA for a hint.)

5. **From Brockett[70]**

   Suppose that the boundary conditions for $\dot{x} = A(t)x(t)$ were specified in part at t_0 and in part at t_1. In particular suppose that

$$Mx(t_0) + Nx(t_1) = b$$

   with Rank $(M, N) = n$ (dimension of x). Show that this *two point boundary value problem* has a unique solution if $M + N\Phi(t_1, t_0)$ is nonsingular.

6. **H dot or Lax Equation** Let $\Omega \in \mathbb{R}^{n\times n}$ be a skew symmetric matrix and $H(0) \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Show that the differential equation

$$\dot{H} = [H, \Omega] \qquad H(t) \in \mathbb{R}^{n\times n}$$

   generates a flow on symmetric matrices, that is $H(t)$ is symmetric for all t. Also show that the eigenvalues of $H(t)$ are the eigenvalues of $H(0)$. These flows are called *iso-spectral*.