Problem Set 4
EE221a: Linear Systems Theory

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1. A conservative physical system is modeled by $\dot{x} = Ax, A \in \mathbb{R}^{n \times n}$ and it is normalized so that along any trajectory $t \rightarrow ||x(t)||^2$ is constant (a measure of energy).
   - What can you say about the eigenvalues of A ?
   - To integrate this differential equation numerically, we have a choice of three methods:
     - **Forward Euler** : $\xi_{k+1} = (I + hA)\xi_k, \;\; \xi_0 = x(0)$.
     - **Backward Euler** : $\xi_{k+1} = (I - hA)^{-1}\xi_k, \;\; \xi_0 = x(0)$.
     - **Combination Backward and Forward Euler** : $\xi_{k+1} = (I + \frac{h}{2}A)(I - \frac{h}{2}A)^{-1}\xi_k, \quad \xi_0 = x(0)$.

     We would like to choose a step size $h < 2\text{min}_i|\lambda_i(A)|$. Select the method which is appropriate to this problem and justify your choice.

2. If $(\lambda_i)_{i=1}^{n}$ are the eigenvalues of A, (assumed to be distinct) find the eigenvalues and eigenvectors of the linear maps:
   i) $L_1 : P \in \mathbb{R}^{n \times n} \rightarrow A^T P - PA \in \mathbb{R}^{n \times n}$.
   ii) $L_2 : P \in \mathbb{R}^{n \times n} \rightarrow A^T PA - P \in \mathbb{R}^{n \times n}$.

3. **Stiff Differential Equations**

   In the simulation of several engineering systems we encounter parasitic elements which result in the differential equation becoming "stiff". For example, parasitic capacitances and inductances in electronic circuits. This results in some state variables changing much more rapidly than the others. To represent this, consider the system with x_1 representing the "slow" variables and x_2 the "fast" variables.

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 \\ \epsilon\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 \end{aligned}$$

   with $x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}^m$ and A_{22} non-singular. Show that m eigenvalues go to ∞ like $\frac{\sigma(A_{22})}{\epsilon}$ and the other n tend to $\sigma(A_{11} - A_{12}A_{22}^{-1}A_{21})$. In circuit theory, we refer to the system

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 \\ 0 &= A_{21}x_1 + A_{22}x_2 \end{aligned}$$

   as the *singularly perturbed* or *low frequency approximation.*

   In electronic circuits, we also have in addition to parasitic (small) capacitances, coupling (large) capacitances. These are modeled by

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \\ \epsilon\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \\ \mu\dot{x}_3 &= A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \end{aligned}$$

with $\epsilon > 0$ small and $\mu > 0$ large. A *mid frequency model* takes $\epsilon = 0, \mu = \infty$, a *low frequency model* takes $\epsilon = 0$ and sets $\mu = \infty$ in the $\tau = \frac{t}{\mu}$ time scale and a *high frequency model* sets $\mu = \infty$ and then sets $\epsilon = 0$ in the time scale $\tau = \frac{t}{\epsilon}$. Find the relationship between the eigenvalues of each if these models.

4. Consider the control system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

   Find some input $u(t) : [0, 1] \to \mathbb{R}$ which takes the zero state to $(1, 1)^T$ at time 1. (Hint: try $u(t) = a_1 e^t + a_2 e^{2t}$).

5. Consider the standard, square, i.e. $(n_i = n_o)$ system representation A, B, C, D. Assume that A has distinct eigenvalues with eigenvlaues $\lambda_i, i = 1, \ldots, n$ and right, left eigenvectors e_i, η_i with $\eta_i^T e_j = \delta_{ij}$ for all $i, j = 1, \ldots, n$. The transfer function $\hat{H}(s) := C(sI - A)^{-1}B + D$. Show that p is a pole of $\hat{H}(s)$ $\Leftrightarrow$ (i) $p = \lambda_i$ for some i, (ii) $Ce_i \neq \theta_{n_0}$ and (iii) $\eta_i^T B \neq \theta_{n_i}$. In class, we have shown that $z \in \mathbb{C}$ is a zero of $\hat{H}(s)$, if there exists $x_0 \in \mathbb{R}^n, u_0 \in \mathbb{R}^{n_i}$ such that the system response starting from x_0 at time 0 is $y(t) \equiv 0$ for all $t \geq 0$ for an input $u(t) = u_0 e^{zt}$. Characterize x_0 in terms of A, B, C, D. For initial condition $x \neq x_0$, find a formula for the output response for the same input $u_0 e^{zt}$.

6. Given a collection of n linearly independent eigenvectors: $x_1, x_2, \ldots, x_n \in V$ define

$$\begin{aligned} v_1 &= x_1(||x_1||)^{-1} \\ v_2 &= (x_2 - v_1 < v_1, x_2 >)(||x_2 - v_1 < v_1, x_2 > ||)^{-1} \\ \vdots &\quad \vdots \end{aligned}$$

   Complete the recursion for v_3, v_4, etc. and show that the v_i are an orthonormal set.

7. You are given a linear map L from a Hilbert space $H_1 \to H_2$. You do not know if L is either injective or surjective. Given $y \in H_2$, find the $x^* \in H_1$ of *least norm* which minimizes the error

$$||y - Lx||^2$$

8. Let $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{k \times p}, D \in \mathbb{C}^{m \times p}$. Now when can you solve the following equation

$$AXB = D$$

   for $X \in \mathbb{C}^{n \times k}$. When is the solution unique?