Solution Set 1: EECS 222

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1. One way of studying a differential equation is by finding first integrals of motion. A function \( H : \mathbb{R}^2 \to \mathbb{R} \) is called a first integral for the planar dynamical system \( \dot{x} = f(x) \) if

\[
\dot{H} = 0,
\]

where \( \dot{H} : \mathbb{R}^2 \to \mathbb{R} \) is the orbital derivative of \( H \). It is defined by

\[
\dot{H}(x(t)) = \left. \frac{d}{dt} \right|_{t=t} H(x(t)),
\]

where \( x(t) \) is the solution of \( \dot{x} = f(x) \) starting at \( x_0 \) at time \( t = 0 \). Therefore, \( H \) is a first integral if it is constant along trajectories.

Find a first integral for

\[
\ddot{\theta} = a - b \sin \theta,
\]

where \( \theta \in \mathbb{R} \), and use it to draw an approximate phase portrait. You may assume that \( a, b > 0 \) and \( a \neq b \).

**Solution.** Setting \( x_1 = \theta \), and \( \dot{x}_1 = x_2 \) yields the following system:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= a - b \sin x_1.
\end{align*}
\]

We seek \( H(x_1, x_2) \) in the form \( H_1(x_1) + H_2(x_2) \); \( H_1 \) and \( H_2 \) need to satisfy

\[
\dot{H}(x) = \text{grad } H(x) \cdot f(x) = H'_1(x_1)x_2 + H'_2(x_2)(a - b \sin x_1) = 0,
\]

where \( \text{grad } H(x) \) is the gradient of \( H \) at \( x \) and \( f(x) = (x_2, a - b \sin x_1)^T \). It is not difficult to see that we can take:

\[
H(x_1, x_2) = \frac{1}{2} x_2^2 - (ax_1 + b \cos x_1).
\]

We have the following two cases:
Case 1: $a > b$.

Then $a - b \sin x_1 > 0$, so there are no equilibria. Trajectories of the above system are the level curves

$$H(x_1, x_2) = \text{constant}.$$ 

Observe also that along the $x_1$-axis, the vector field is perpendicular to that axis and points in the positive $x_2$-direction. For the phase portrait, see Figure 1.

![Figure 1: Case 1, $a = 2, b = 1$.](image)

Case 2: $a < b$.

Equilibrium points are $(x_{2n}, 0)$ and $(x_{2n+1}, 0)$, where $x_{2n} = \gamma + 2n\pi$, and $x_{2n+1} = -\gamma + (2n + 1)\pi$, $\gamma = \arcsin(a/b)$ and $n$ is an integer. The linear part of the system at $(x_n, 0)$ is

$$\begin{bmatrix} 0 & 1 \\ -b \cos x_n & 0 \end{bmatrix}.$$ 

The eigenvalues $\lambda_n$ satisfy $\lambda_n^2 + b \cos x_n = 0$, therefore:

- $\lambda_{2n} = \pm i(b^2 - a^2)^{1/4}$, so $(x_{2n}, 0)$ are non-hyperbolic equilibria and we cannot use Hartman-Grobman’s theorem. However, by studying the function $\phi(x) = H(x, 0)$ (i.e. the restriction of $H$ to the $x_1$-axis), it can be shown that for $x$ sufficiently close to $x_{2n}$, there exists $\tilde{x} \neq x$ such that $\phi(x) = \phi(\tilde{x})$, that is, we have a closed orbit of $f$. This shows that each equilibrium $(x_{2n}, 0)$ is surrounded by a continuum of closed orbits.

- $\lambda_{2n+1} = \pm (b^2 - a^2)^{1/4}$, so by Hartman-Grobman, $(x_{2n+1}, 0)$ are saddles.

Note that the vector field $f$, hence the phase portrait, is $2\pi$-periodic in $x_2$. From this and previous observations, we obtain the phase portrait in Fig. 2.
2. Consider the Volterra-Lotka predator-prey model,
\[
\begin{align*}
\dot{x} &= (A - B y - \lambda x)x \\
\dot{y} &= (C x - D - \mu y)y,
\end{align*}
\]
where \(x\) is the prey population, \(y\) the predator population, \(A, B, C, D, \lambda, \mu > 0\) and \(x, y \geq 0\).

(a) Find the equilibria of the system. How do they depend on the parameters \(A, B, C, D, \lambda, \mu\)?

(b) Simulate the behavior of the system for different values of the parameters. Discover and draw all possible qualitatively different phase portraits, depending on the relationship among \(A, B, C, D, \lambda, \mu\) found in (a). Justify your solution.

Solution. Divide the first quadrant \((x, y \geq 0)\) into sectors by two lines
\[
L : A - B y - \lambda x = 0,
\]
\[
M : C x - D - \mu y = 0.
\]
Along these lines \(\dot{x} = 0\) and \(\dot{y} = 0\). There are two possibilities, depending on whether \(L\) and \(M\) intersect in the first quadrant or not. Note that \(L\) and \(M\) meet the \(x\)-axis at \((A/\lambda, 0)\) and \((D/C, 0)\) respectively.

Case 1: \(L\) and \(M\) do not intersect, i.e., \(\frac{A}{\lambda} < \frac{D}{C}\) (Fig. 3).

In this case, we claim that every trajectory \((x(t), y(t))\) (starting from inside the first quadrant) approaches the point \((A/\lambda, 0)\) (i.e., prey \(x\) approaches the value \(A/\lambda\) and the predators \(y\) die out). This is because it is impossible to have \(\dot{x} > 0\) and \(\dot{y} > 0\) (i.e. predators and prey cannot both increase) at the same time. By analyzing the vector field (more precisely: by looking at the sign of \(\dot{x}\) and \(\dot{y}\) in each sector), we see that if \(x > A/\lambda\), \(x\) must decrease and after a while \(y\) also starts to decrease (when the trajectory crosses \(M\)). After that point, \(x\), the prey, can never increase past \(A/\lambda\), so \(y\), the predators, continue to decrease. If the trajectory crosses \(L\), the prey \(x\) increases again (but not past \(A/\lambda\)), while the predator \(y\) continues to die off. In the limit, \((x, y) \to (A/\lambda, 0)\).
Case 2: \( L \) and \( M \) intersect, i.e. \( A/\lambda > D/C \)

Let \( z = (a, b) \) be the intersection of \( L \) and \( M \); this point is clearly an equilibrium. The linearization at \( z \) is

\[
\begin{bmatrix}
-\lambda a & -Ba \\
Cb & -\mu b
\end{bmatrix},
\]

which has both eigenvalues with negative real parts. Therefore, \( z \) is a sink (i.e., asymptotically stable).

In addition to \( z \) and \((0,0)\), there is also an equilibrium at \( w = (A/\lambda,0) \), where \( L \) meets the \( x \)-axis. It is easy to verify that this is a saddle.

Let \( S \) be the rectangle whose corners are \((0,0)\), \((p,0)\), \((p,q)\) and \((0,q)\), where \( p,q \) are chosen so that \( p > A/\lambda \) and \( q > A/B \) (observe that \( L \) meets the \( y \)-axis at \((0,A/B)\)). Every trajectory at a boundary point of \( S \) enters \( S \) or is a part of the boundary. Therefore, \( S \) is positively invariant (see Fig. 4).

By the Poincaré-Bendixson theorem, the \( \omega \)-limit set of any point \((x,y)\) in \( S \) with \( x > 0 \), \( y > 0 \), must be a limit cycle or one of the three equilibria \((0,0)\), \( z \) or \( w \) (observe that there are no saddle connections). It is not difficult to rule out \((0,0)\) and \((A/\lambda,0)\) (\( x \) increases near \((0,0)\) and \( y \) increases near \((A/\lambda,0)\)). Therefore, \( \omega(x,y) \) is either \( z \) (Fig. 5) or a limit cycle which must surround \( z \) (Fig. 6).

Observe that every trajectory \((x(t),y(t))\), for any initial values, eventually enters \( S \), i.e. \( x(t) < p \) and \( y(t) < q \), for \( t \geq t_0 \). This shows that in the long run, a trajectory either approaches \( z \) or spirals down to a limit cycle, or: predators and prey settle down either to a constant or periodic population.

3. (The Logistic Map) Let \( f_\lambda(x) = \lambda x(1-x) \), and consider the discrete time dynamical system obtained by iterating \( f_\lambda : \mathbb{R} \to \mathbb{R} \).

(a) For \( 0 < \lambda < 1 \), show that the only equilibrium is 0 and that \( f_\lambda^n(x) \to 0 \), as \( n \to \infty \), for all \( x \in [0,1] \).
Figure 4: Predator-prey equation for $\frac{A}{\lambda} < \frac{D}{C}$.

$$x' = (2 - y - x) x$$
$$y' = (10 x - 10 - y) y$$

Figure 5: Volterra-Lotka, Case 2 without a limit cycle.
For $1 < \lambda < 3$, find all equilibria and classify them into stable (attracting) and unstable (repelling). Show also that if $x < 0$ or $x > 1$, then \( f^n_\lambda(x) \to -\infty \), as $n \to \infty$.

For $1 < \lambda < 3$, let $x_\lambda = (\lambda - 1)/\lambda$. Show that there exists an interval $J \subset (0, 1)$ such that for all $x \in J$, $f^n_\lambda(x) \to x_\lambda$, as $n \to \infty$. ($J$ is called the domain of attraction of $x_\lambda$.) Compute this interval as best as you can.

**Solution.**

(a) If $0 < \lambda < 1$, then the only fixed point is 0 and it is stable because $|f'_\lambda(0)| = \lambda < 1$.

(b) Assume $1 < \lambda < 3$. The only two equilibria are 0 and $x_\lambda = (\lambda - 1)/\lambda$. Since $|f'_\lambda(0)| = \lambda > 1$ and $|f'_\lambda(x_\lambda)| = |\lambda - 2| < 1$, 0 is unstable and $x_\lambda$ is stable.

Suppose $x < 0$. Then we have $f'_\lambda(x) > 1$, and by the Mean Value Theorem (MVT):

\[
0 > x > f_\lambda(x) > f^2_\lambda(x) > \ldots .
\]

If this sequence were bounded, it would converge to a negative fixed point of $f_\lambda$ which does not exist. Therefore, $f^n_\lambda(x) \to -\infty$.

If $x > 1$, then $f_\lambda(x) < 0$, so $f^n_\lambda(x) = f^{n-1}_\lambda(f_\lambda(x)) \to -\infty$. 

Figure 6: Volterra-Lotka, Case 2 with a limit cycle.
(c) We claim: $J = (0, 1)$, where the assumption is still that $1 < \lambda < 3$.

First assume $1 < \lambda < 2$. If $x \in (0, \frac{1}{2})$, then a calculation (not involving the MVT) shows:

$$|f_{\lambda}(x) - x_{\lambda}| < |x - x_{\lambda}|.$$  

Since $0 < f_{\lambda}(x) < 1/2$, by induction we get $f^n_{\lambda}(x) \to x_{\lambda}$, as $n \to \infty$.

If $0 < x < 1$, then $f_{\lambda}(x) \in (0, \frac{1}{2})$, and the preceding argument can be applied.

Now assume $2 \leq \lambda < 3$. Since $|f'_{\lambda}(x_{\lambda})| = |2 - \lambda| < 1$, there exists an interval $A$ around $x_{\lambda}$ on which $|f^n_{\lambda}| < 1$. It can be shown that:

- $A = (\frac{1}{2} - \frac{1}{2x}, \frac{1}{2} + \frac{1}{2x})$;
- For every $x \in (0, 1)$ there exists $n \geq 0$ such that $f^n_{\lambda}(x) \in A$.

4. Show that the circle $r = \sqrt{\lambda}$ is an attracting set (i.e., all orbits starting in some neighborhood of the set converge to the set) for the dynamical system (in polar coordinates)

$$\dot{r} = \lambda r - r^3, \quad \dot{\theta} = 1 - \cos 2\theta.$$  

Which of the equilibrium points are attractors and which repellors? Describe the $\alpha$- and $\omega$-limit sets (sets of limit points as time goes to $-\infty$ and $+\infty$, respectively) for typical points inside and outside the circle $r = 1$ and in the upper and lower half planes.

**Solution.** If $\lambda < 0$, there is a single stable equilibrium point at the origin, since $\dot{r} < 0$.

Since $r = \sqrt{\lambda}$ is a stable ($\dot{r} > 0$ if $0 < r < 1$, and $\dot{r} < 0$ if $r > 1$) equilibrium for $\dot{r} = \lambda r - r^3$, it follows that the circle $C: r = \sqrt{\lambda}$ is an invariant, attracting set for the given system (S). This holds for $\lambda > 0$.

The only equilibria of (S) are (in polar coordinates) $p_1 = (\sqrt{\lambda}, 0)$ and $p_2 = (\sqrt{\lambda}, \pi)$, that is, the points $(1, 0)$ and $(-1, 0)$ in rectangular coordinates. Observe that $A = \text{the } x\text{-axis minus}$

the origin (at which the system is not defined) is invariant, since $\dot{\theta} = 0$ on it. Also, $p_1$ and $p_2$ attract points from the corresponding component of $A$.

Let $p$ be a point in the upper half plane for which $r > 1$. Then, since $\dot{\theta} > 0$ (except on $A$) and $\dot{r} < 0$, the orbit of $p$ must converge to $p_2$. Thus $\omega(p) = \{p_2\}$. It is not difficult to see that $\alpha(p)$ is empty, since the negative orbit of $p$ is unbounded.

Let $p$ be a point in the upper half plane for which $r < \sqrt{\lambda}$. A similar argument to the previous one shows that $\omega(p) = \{p_2\}$, but now $\alpha(p) = \{0\}$.

Analogously, if $p$ is in the lower half plane and $r > \sqrt{\lambda}$, then $\omega(p) = \{p_1\}$ and $\alpha(p)$ is empty. If $r < \sqrt{\lambda}$, then $\omega(p) = \{p_1\}$ and $\alpha(p) = \{0\}$.

Therefore, neither equilibrium is an attractor nor repellor. Based on these observations, we can now sketch the phase portrait of (S) drawn for $\lambda = 1$; see Fig. 7 (which was obtained by putting (S) into rectangular coordinates).

5. (Gradient systems) A dynamical system $\dot{x} = F(x)$, where $F(x) = -\text{grad } V(x)$ (minus the gradient of a smooth function $V$), is called a gradient system.

(a) If $x(t)$ is a trajectory of a gradient system $\dot{x} = -\text{grad } V(x)$, show that

$$\frac{d}{dt} V(x(t)) = -|\text{grad } V(x(t))|^2.$$
Figure 7: Phase portrait in problem 4.

(b) If the Hessian $D^2V$ of $V$ (i.e., the matrix of second partials of $V$) is nonsingular at the equilibria of $\dot{x} = -\nabla V(x)$, show that the equilibria cannot be spirals or centers.

Solution. (a) We have:

\[
\frac{d}{dt} V(x(t)) = D V(x(t)) \dot{x}(t) = \nabla V(x(t)) \cdot \dot{x}(t) = -\nabla V(x(t)) \cdot \nabla V(x(t)) = -|\nabla V(x(t))|^2.
\]

Therefore, $V$ is strictly decreasing on regular (i.e., non-stationary) orbits so gradient systems do not have closed orbits.

(b) Let $F = -\nabla V$. Then

\[ DF(x) = -D^2V(x), \]

the Hessian of $V$. Since $V$ is of class $C^2$, its mixed partials are the same, so $D^2V(x)$ is a symmetric matrix. Therefore, its eigenvalues must be real numbers. If the Hessian of $V$ is everywhere nonsingular, then all equilibria are hyperbolic. Thus in dimension two we can only have saddles, stable and unstable nodes, but no spirals or centers.

Remark. In general, if we allow the Hessian to be singular at critical points of $V$, (a) still excludes centers, but the proof that in dimension two there are no spirals is more complicated.
Here is a version of the proof (due to Pablo Anigstein who took this class in 2000).

Suppose there is a spiral towards an equilibrium $p$ of a gradient system. Then there exists a smooth curve $S$ starting at $p$ which is perpendicular to the vector field $F = -\text{grad } V = (f_1, f_2)^T$. Let $\Gamma$ be a closed curve consisting of an arc $x_1 x_2$ of the spiral and a segment of $S$ so that $x_2$ is the first intersection of the (spiral) orbit through $x_1$ with $S$. Then by the Jordan Curve Theorem, $\Gamma$ bounds a disk $D$ (see Fig. 8). By Green’s theorem:

$$
\int_{\Gamma} f_1 dx_1 + f_2 dx_2 = \int_D \left( \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) dx_1 dx_2
$$

$$
= \int_D \left( \frac{\partial^2 V}{\partial x_1 \partial x_2} - \frac{\partial^2 V}{\partial x_2 \partial x_1} \right) dx_1 dx_2
$$

$$
= 0.
$$

But the integral of $F$ along $\Gamma$ can be decomposed into

$$
\int_{\Gamma} f_1 dx_1 + f_2 dx_2 = \left( \int_{x_1 x_2; \text{along the spiral}} + \int_{x_1 x_2; \text{along } S} \right) f_1 dx_1 + f_2 dx_2,
$$

where the second one is 0 but the first one must be strictly negative. Contradiction! Therefore, there are no spirals.