1. Draw the phase portrait of a reaction-diffusion system

\[
\dot{x}_1 = 2(x_2 - x_1) + x_1(1 - x_1^2) \\
\dot{x}_2 = -2(x_2 - x_1) + x_2(1 - x_2^2).
\]

List the equilibria and their types. Does the system have limit cycles? (Hint: Show that this is a gradient system.)

**Solution.** By solving \(\dot{x}_1 = 0, \dot{x}_2 = 0\), we get that the system has three equilibria: \(p_0 = (0, 0)\), \(p_1 = (1, 1)\) and \(p_2 = (-1, -1)\). Linearizing the system at the equilibria, it is easy to see that \(p_0\) is a saddle, while \(p_1\) and \(p_2\) are stable nodes. The unstable and stable spaces of \(p_0\) are given by: \(x_1 = x_2 \ (|x_1| < 1)\) and \(x_1 + x_2 = 0\), respectively.

In fact, if we denote the vector field of the system by \(f = (f_1, f_2)^T\), then

\[
\frac{\partial f_1}{\partial x_2} = 2 = \frac{\partial f_2}{\partial x_1},
\]

which implies that \(f\) is a gradient vector field. By integrating \(f_1\) relative to \(x_1\), differentiating the obtained function with respect to \(x_2\) and equating the result with \(f_2\), we get that \(f = -\nabla V\), where

\[
V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{4}(x_1^4 + x_2^4) - 2x_1x_2.
\]

Therefore, by exercise 5. from Homework 1, the system has no closed orbits. (Alternatively, one could observe that the divergence of \(f\) is \(-2 - 3(x_1^2 + x_2^2)\) and use Bendixson’s theorem.)
The phase portrait of $f$ is shown in Fig. 1.

2. Shown below are phase portraits of some vector fields in the plane. Determine which of them are correct and which are incorrect. Modify the incorrect ones, not by deleting any orbits, but by changing their stability type or adding new orbits.

**Solution.** Phase portrait (a) is correct, but here’s how the given trajectories can be extended.

(c) Let $P$ be the region bounded by four saddle connections. We claim that $P$ must contain an equilibrium. Since $P$ is invariant, for any $p \in P$, $\omega(p)$ and $\alpha(p)$ can be: (i) an equilibrium, or: (ii) a limit cycle or: (iii) a finite connected union of saddle connections. If (ii), then the region bounded by the limit cycle must contain an equilibrium. If (iii), apply the same reasoning to the region $Q$ bounded by these saddle connections (if $Q = P$, reverse the time).
3. Draw phase portraits of the following 1-dimensional systems as $\mu$ changes:

(a) $\dot{x} = \mu^2x - x^3$

(b) $\dot{x} = \mu^2\alpha x + 2\mu x^3 - x^5$, for different $\alpha$.

**Solution.** (a) Let $f_{\mu}(x) = \mu^2x - x^3$. The equilibria are $x_0(\mu) = 0$, $x_1(\mu) = \mu$ and $x_2(\mu) = -\mu$.

By looking at $f_{\mu}'(x_i(\mu))$ for $i = 0, 1, 2$, we see that $\mu = 0$ is the bifurcation value and obtain the following bifurcation diagram (Fig. 2).

![Figure 2: Problem 3.(a)](image)

(b) Let $f_{\mu}(x) = \alpha\mu^2x + 2\mu x^3 - x^5$. The solutions to $f_{\mu}(x) = 0$ are: $x_1(\mu) = 0$, $x_2(\mu)$, $x_3(\mu)$, $x_4(\mu) = -x_2(\mu)$ and $x_5(\mu) = -x_3(\mu)$, where:

$$x_2(\mu) = \sqrt{\mu + |\mu|\sqrt{1 + \alpha}}, \quad x_3(\mu) = \sqrt{\mu - |\mu|\sqrt{1 + \alpha}}.$$  

We also have:

$$f_{\mu}'(x) = \alpha\mu^2 + 6\mu x^2 - 5x^4.$$  

To investigate stability of equilibria of $\dot{x} = f_{\mu}(x)$, we need to consider the following cases:

**Case 1:** $\alpha < -1$ (Fig. 3).

The only equilibrium is $x_1(\mu)$ and since $f_{\mu}'(x_1(\mu)) = \alpha\mu^2$, it is stable for $\mu \neq 0$. If $\mu = 0$, then $f_{\mu}(x) = -x^5$, which also has a stable equilibrium at 0.

**Case 2:** $-1 < \alpha < 0$.

The equilibria are $x_1(\mu), \ldots, x_5(\mu)$, if $\mu \geq 0$, and only $x_1(\mu)$ if $\mu < 0$. By checking the sign of $f_{\mu}'(x_i(\mu))$, for $i = 1, \ldots, 4$, we obtain:

- if $\mu \leq 0$, the only equilibrium $x_1(\mu) = 0$ is stable;
- if $\mu > 0$, then: $x_1(\mu), x_2(\mu), x_4(\mu)$ are all stable, and $x_3(\mu), x_5(\mu)$ are unstable.

**Case 3:** $\alpha = -1$ or $\alpha = 0$ (Fig. 5).
Figure 3: Problem 2(b), Case 1.

Figure 4: Problem 2(b), Case 2.

Figure 5: Problem 2(b), Case 3, cases $\alpha = -1$ (left) and $\alpha = 0$ (right).
Case 4: $\alpha > 0$.

The only equilibria are $x_1(\mu), x_2(\mu)$ and $x_4(\mu) = -x_2(\mu)$, for all $\mu \in \mathbb{R}$. By checking the sign of $f_\mu'$ at these points we obtain that $x_1(\mu)$ is unstable, while the other ones are stable for all $\mu$ (Fig. 6).

![Figure 6: Problem 2(b), Case 4.](image)

4. Let $f$ be a smooth vector field on the annulus $A = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$.

Assume $f$ points inward along the boundary of $A$, and that for every $0 \leq \alpha \leq 2\pi$, the radial segment (in polar coordinates)

$$S_\alpha = \{(r, \theta) : 1 \leq r \leq 2, \ \theta = \alpha\}$$

is a local cross section in the sense defined in class; that is, at every point $p \in S_\alpha$, the angle between $f(p)$ and $S_\alpha$ is not zero.

(a) Let $p \in S_0$ be an arbitrary point. Show that the orbit of $f$ starting at $p$ returns to $S_0$ after some positive time.

(b) Show that every continuous function $\phi : [1, 2] \to [1, 2]$ has a fixed point.

(c) For $p \in S_0$, let $\psi(p)$ be the point of first return to $S_0$ of the orbit starting at $p$. Show that $\psi : S_0 \to S_0$ has a fixed point.

(d) Show that there exists a closed orbit of $f$ in $A$.

**Solution.** (a) First of all, observe that since $f$ points inward along the boundary of $A$, $A$ is positively invariant relative to the flow $\Phi_t$ of $f$. Let $\Theta$ be the angular coordinate in the polar coordinate system on $A$. Let $p = (r, \theta) \in A$ be an arbitrary point and assume it belongs to $S_\alpha$, for some $\alpha$. Then by assumption that $S_\alpha$ is a local cross section for the flow $\Phi_t$, we obtain

$$\frac{d}{dt} \Theta(\Phi_t(r, \theta)) \neq 0.$$ 

Since both $\Theta$ and $\Phi_t$ are smooth on $A$, the left hand side is either always $> 0$ or always $< 0$. Assume the former. Since $A$ is compact, $\frac{d}{dt} \Theta(\Phi_t(r, \theta))$ has a minimum $m > 0$ on $A$. By the Fundamental Theorem of Calculus,

$$\Theta(\Phi_t(r, \theta)) - \Theta(r, \theta) \geq mt$$

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which becomes greater than $2\pi$ for sufficiently large $t > 0$. Therefore, the orbit of $f$ starting at $p \in S_0$ returns to $S_0$ after some positive time.

(b) If $\phi(1) = 1$ or $\phi(2) = 2$, there is nothing to prove. So assume $\phi(1) > 1$ and $\phi(2) < 2$, and define $g(s) = \phi(s) - s$. Then $g$ is continuous, $g(1) > 0$, and $g(2) < 0$. Therefore, by the Intermediate Value Theorem, there exists a point $s_0 \in (1, 2)$ such that $g(s_0) = 0$. Clearly, $s_0$ is a fixed point for $\phi$.

(c) Observe that by the Implicit Function Theorem $\psi$ is a smooth, hence continuous function. Therefore, since $S_0 = [1, 2] \times \{0\} \approx [1, 2]$, we can take $\psi = \phi$ and use part (b).

(d) Let $p \in S_0$ be a fixed point of $\psi$ from (c). Then, by definition of $\psi$, $p = \psi(p) = \Phi_\tau(p)$ for some $\tau > 0$, i.e., the orbit of $p$ is closed. ■

5. Consider the following parametrized family of differential equations:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= \mu_1 x + \mu_2 y + x^3 - x^2 y.
\end{align*}
\]

(a) What does the flow near $(0, 0)$ look like when $\mu_1 = \mu_2 = 0$?

(b) What bifurcations occur at $\mu_1 = 0$ and at $\mu_2 = 0$ (for $\mu_1 < 0$)?

(c) Use Bendixson’s theorem and index theory to rule out parameter regions where there are no periodic orbits.

(d) Conjecture the full bifurcation diagram.

Solution. (a) For the phase portrait when $\mu_1 = \mu_2 = 0$, see Fig. 7.

(b) The equilibria of the system are:

\[
\begin{align*}
p_0(\mu) &= (0, 0), \quad p_1(\mu) = (\sqrt{-\mu_1}, 0), \quad p_2(\mu) = (-\sqrt{-\mu_1}, 0),
\end{align*}
\]

where $\mu = (\mu_1, \mu_2)$.

At $p_0(\mu)$, the eigenvalues of the linearization are $[\mu_2 \pm \sqrt{\mu_2^2 + 4\mu_1}] / 2$, so:

- $\mu_1 < -\frac{\mu_2^2}{4}$: if $\mu_2 > 0$, then $p_0(\mu)$ is a source (Fig. 8), and if $\mu_2 < 0$, it is a sink (Fig. 9);

- $-\frac{\mu_2^2}{4} < \mu_1 < 0$: if $\mu_2 < 0$, then $p_0(\mu)$ is a stable node (Fig. 10), and if $\mu_2 > 0$, then it is an unstable node (Fig. 11);

- if $\mu_1 > 0$ and $\mu_2 \in \mathbb{R}$, then $p_0(\mu)$ is a saddle.

At $p_{1,2}(\mu)$, the eigenvalues of the linearization are:

\[
\frac{\mu_1 + \mu_2 \pm \sqrt{(\mu_1 + \mu_2)^2 - 8\mu_1}}{2}
\]
Figure 7: The phase portrait in problem 5 for $\mu_1 = \mu_2 = 0$.

Figure 8: The phase portrait in problem 5(b) for $\mu_1 < -\frac{\mu_2}{4}$ and $\mu_2 > 0$. 
Figure 9: The phase portrait in problem 5(b) for $\mu_1 < -\frac{\mu_2^2}{4}$ and $\mu_2 < 0$.

Figure 10: The phase portrait in problem 5(b) for $-\frac{\mu_2^2}{4} < \mu_1 < 0$ and $\mu_2 < 0$. 
Since $\mu_1 < 0$, both $p_1(\mu)$ and $p_2(\mu)$ are saddles.

To summarize, at $\mu_1 = 0$, when $\mu_2 > 0$, two saddles $(\pm \sqrt{-\mu}, 0)$ and the unstable equilibrium point $(0, 0)$ merge into the unstable equilibrium $(0, 0)$. At $\mu_1 = 0$, when $\mu_2 < 0$, two saddles $(\pm \sqrt{-\mu}, 0)$ and the stable equilibrium point $(0, 0)$ merge into the unstable equilibrium $(0, 0)$, i.e. a pitchfork bifurcation appears.

At $\mu_2 = 0$, when $\mu_1 < 0$, the eigenvalue of the linearization at $(0, 0)$ cross the $jw$-axis (i.e. from $C_0^-$ when $\mu_2 < 0$ to $C_0^+$ when $\mu_2 > 0$) and not through the origin. In fact, a stable limit cycle and an unstable equilibrium arise from a stable equilibrium, i.e. a (supercritical) Hopf bifurcation appears.

(c) Since the divergence of $f$ is $\text{div } f = \mu_2 - x^2$, we get that $\text{div } f \leq 0$ for $\mu_2 \leq 0$, so $f$ has no closed orbits when $\mu_2 \leq 0$. It is not difficult to check that if $\mu_1 \geq 0$, the only equilibrium, namely, $0$, is a saddle, hence has index $-1$, which guarantees nonexistence of closed orbits (recall that the index of a smooth vector field on the disk bounded by a closed orbit is 1). Therefore, $f$ has no closed orbits when $\mu_1 \geq 0$ or $\mu_2 \leq 0$. 

Figure 11: The phase portrait in problem 5(b) for $-\mu_1/4 < \mu_1 < 0$ and $\mu_2 > 0$. 
6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism; in other words, $f$ is invertible, and both $f$ and $f^{-1}$ are smooth maps. Let $p_0, p_1, \ldots, p_{N-1}$ be a period $N$ cycle of $f$; that is, $p_k = f^k(p_0)$ for $1 \leq k \leq N-1$ and $f^N(p_0) = p_0$. Prove that the matrices $Df^N(p_k)$ have the same spectrum for $k = 0, \ldots, N-1$.

**Solution.** By the Chain Rule,

$$Df^N(p_k) = A_{k-1}A_{k-2} \cdots A_0A_{N-1}A_{N-2} \cdots A_{k+1}A_k,$$

where $A_i = Df(p_i)$. Since $f$ is a diffeomorphism, each matrix $A_i$ is invertible, so we get

$$Df^N(p_{k-1}) = A_{k-1}^{-1}Df^N(p_k)A_{k-1}.$$

Therefore, for all $k = 1, 2, \ldots, N$, $Df^N(p_{k-1})$ and $Df^N(p_k)$ are similar, thus have the same spectrum. $\blacksquare$