University of California, Berkeley
Department of EECS

Instructor: S. Sastry | Spring 2007
Assigned: February 15 | Due: **March 1**

EE222: Homework Assignment 3

1. (Perturbations of dynamical systems) Let $U \subset \mathbb{R}^n$ be an open set and $f, g : U \to \mathbb{R}^n$ smooth vector fields on U such that

$$|f(x) - g(x)| < \epsilon,$$

for all $x \in U$. We can think of g as a perturbation of f. Let $0 < \lambda < \infty$ be the Lipschitz constant of f on U. If $x(t)$ and $y(t)$ are solutions to

$$\dot{x} = f(x), \qquad \dot{y} = g(y),$$

respectively, on some interval J containing 0, and $x(0) = y(0)$, show that

$$|x(t) - y(t)| \leq \frac{\epsilon}{\lambda}(e^{\lambda t} - 1),$$

for all $t \in J$. Therefore, if $|f - g|$ is small, then solutions to $\dot{x} = f(x)$ and $\dot{y} = g(y)$, having the same initial values, are close.

2. (The First Variational Equation) Let f be a smooth (at least C^2) vector field on some open set $U \subset \mathbb{R}^n$. Denote by ϕ_t its flow. That is, $\phi_t(x_0)$ is the unique solution to $\dot{x} = f(x)$ such that $x(0) = x_0$. Let $X(t) \stackrel{\text{def}}{=} D\phi_t(x_0)$ be the space derivative (i.e., the Jacobian matrix with respect to x) of ϕ_t at an arbitrary point $x_0 \in U$. Let $A(t) = Df(\phi_t(x_0))$.

(a) Show that $X(t)$ satisfies the following equation, called the First Variational Equation:

$$\dot{X}(t) = A(t)X(t), \qquad X(0) = I,$$

where I is the $n \times n$ identity matrix.

(b) If x_0 is an equilibrium of f, show that

$$X(t) = \exp\left(tDf(x_0)\right),$$

where for a square matrix M, $\exp(M) = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$ is the exponential of M.

3. (Sliding mode control of uncertain linear systems) Consider the linear (time varying) control system:

$$\dot{x} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & 1 \\ -a_1(t) & -a_2(t) & \cdots & -a_n(t) \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u. \tag{1}$$

The $a_i(t)$'s are not known exactly; we only know that $\alpha_i \leq a_i(t) \leq \beta_i$. We also have a bound $|y_d^{(n)}(t)| \leq v$, where $v > 0$ is a constant. The control objective is to get the output of the system $y(t) = x_1(t)$ to track a specified n times differentiable function $y_d(t)$. Show that this can be accomplished by using a discontinuous control law as follows.

Define a surface $S(t) \subset \mathbb{R}^n$ as the level set $s(x,t) = 0$ (with t fixed) of the function

$$s(x,t) = [x_n - y_d^{(n-1)}(t)] + c_2[x_{n-1} - y_d^{(n-2)}(t)] + \cdots + c_n[x_1 - y_d(t)].$$

We choose a control law of the form

$$u = \sum_{i=1}^{n} \gamma_i(x) x_i + \sum_{i=1}^{n-1} k_i(x,t)(x_{i+1} - y_d^{(i)}) - k_n \mathrm{sgn}\, s(x,t),$$

where sgn denotes the sign of a real number (sgn $r = 1$ if $r > 0$, sgn $r = -1$ if $r < 0$).

(a) Choose $\gamma_i(x), k_i(x,t)$ $(1 \leq i \leq n)$ to make $S(t)$ into a *sliding surface*. That is, choose u so that the system (1) satisfies the global sliding condition:

$$\frac{d}{dt} s(x,t)^2 \leq -\epsilon |s(x,t)|, \tag{2}$$

for some $\epsilon > 0$. The derivative is taken along the vector field of (1).

(b) Show that the condition (2) indeed guarantees that every point reaches the sliding surface in finite time.

(c) Show that once on the sliding surface, we can make the tracking error $e(t) = x_1(t) - y_d(t)$ go to zero as $t \to \infty$ by a clever choice of c_i's.

4. (An instability theorem) Let V be a C^1 real-valued function defined on a neighborhood U of an equilibrium $\bar{x}$ of a smooth dynamical system $\dot{x} = f(x)$ in $\mathbb{R}^n$. Suppose $V(\bar{x}) = 0$ and $\dot{V} > 0$ in $U - \{\bar{x}\}$. If $V(x_k) > 0$ for some sequence $x_k \to \bar{x}$, show that $\bar{x}$ is unstable.

5. Let $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ be a continuous function, where $\mathbb{R}^{n \times n}$ is the space of $n \times n$ real matrices. If $X(t)$ is the solution to the initial value problem

$$\dot{X}(t) = A(t)X(t), \qquad X(0) = X_0,$$

show that

$$\det X(t) = (\det X_0) \exp\left[\int_0^t \mathrm{trace}\, A(s)\, ds\right].$$

Recall that for an $n \times n$ matrix $P = (p_{ij})$, trace $P = p_{11} + \cdots + p_{nn}$.