Problem Set on Hybrid Systems

1. Consider the bouncing ball system, given in Figure 1 in the graphical notation.

    1. Derive a hybrid automaton model $BB = (Q, X, f, \mathrm{I}nit, D, E, G, R)$ in the mathematical notation used in the handouts.
    2. Compute the set of states for which continuous evolution is impossible (special care is needed to determine what happens when $x_1 = 0$).
    3. Is BB non-blocking?
    4. Is BB deterministic?
    5. By computing the time interval between two successive bounces, show that the bouncing ball automaton is Zeno.

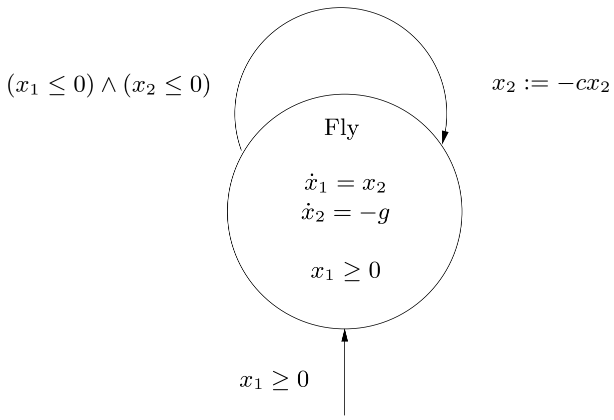

Figure 1: Bouncing ball

2. Consider the thermostat system, shown in Figure 2.

    1. Derive a hybrid automaton model, $Th = (Q, X, f, \mathrm{I}nit, D, E, G, R)$ in the mathematical notation.
    2. Compute the set of states from which continuous evolution is impossible.
    3. Is Th non-blocking?
    4. Is Th deterministic? If not, which of the conditions of the uniqueness Lemma is violated?
    5. Show that the room temperature will always stay in the range $[18, 22]$. What can you infer about the reachable states of Th?
    6. Is the set of states

    $$\{ON, OFF\} \times \{x \in \mathbb{R} \mid x \in [19, 21]\}$$

    invariant?

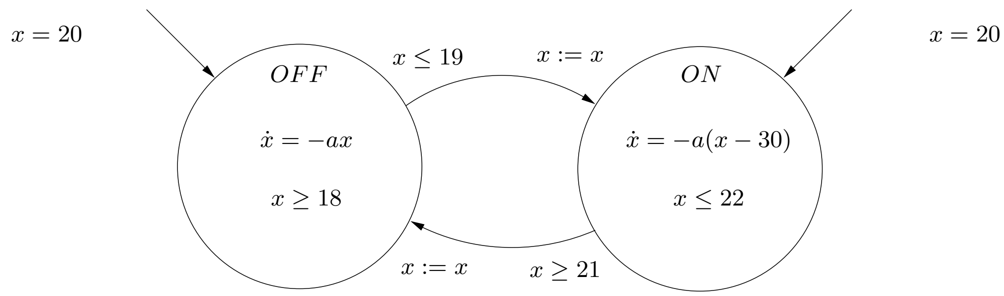

Figure 2: Thermostat system.

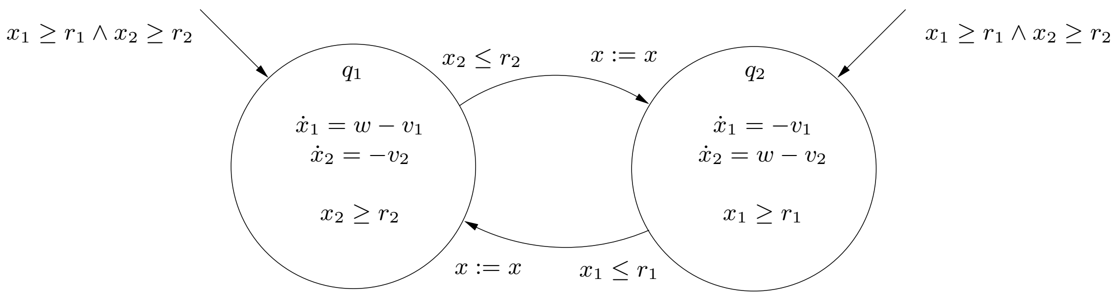

Figure 3: Water tank system.

7. Is the thermostat automaton Zeno?

3. Ursula asked David, a young engineer in her start-up company, to develop a hybrid model of a physical system. David produced a hybrid automaton model using the graphical representation. Looking back over her 4F3 notes from her days at the University of Cambridge, Ursula decided that David's model was useless, since it contained edges ($e \in E$) with empty guards ($G(e) = \emptyset$) or empty resets ($x \in G(e)$, but $R(e, x) = \emptyset$). Now she is threatening to take back David's stock options.

   Show David how he can salvage his model by constructing a hybrid automaton that accepts exactly the same set of executions as the first one, but is such that for all $e \in E$, $G(e) \neq \emptyset$ and for all $x \in G(e)$, $R(e, x) \neq \emptyset$.

4. Consider the water tank system, shown in graphical notation in Figure 3. Assume that $\max\{v_1, v_2\} < w < v_1 + v_2$. Verify that the induction procedure outlined in the notes shows that the set of states $\{(q, x) \mid (x_1 \geq r_1) \wedge (x_2 \geq r_2)\}$ is invariant.

   The condition $\max\{v_1, v_2\} < w < v_1 + v_2$ implies that the rate at which water is added to the system is less than the rate at which water is removed. Physical intuition suggests that in this case at least one of the water tanks will have to eventually drain. Why does the analysis of the hybrid automaton fail to predict that?

5. Having amicably resolved their differences, Ursula and David sit down to a game of "Discrete Controlled Invariance". The game is played on a directed graph (representing a finite state machine, or a finite automaton). The graph for this particular game has 10 states

$Q = \{q_1, \ldots, q_{10}\}$ and is shown in Figure 4. In every state David chooses an action from the set $D = \{1, 2\}$ and Ursula chooses an action from the set $U = \{1, 2\}$. The choices of the two players determine the transition that is to be taken, and therefore the next state. The transitions (edges) of the graph are labelled by a pair (u, d), where u reflects the choice of Ursula and d the choice of David. $*$ denotes a wild card (the action of the particular player is immaterial). Ursula wins the game if she can keep the state of the system in the set $F = \{q_1, \ldots, q_8\}$ while David wins if he can drive the state the complement of F, $F^c = \{q_9, q_{10}\}$.

1. Define a transition function

$$\delta : Q \times U \times D \to Q$$

to encode the dynamics of the game.

2. Using the predecessor operator $Pre_{(u,d)} : P(Q) \to P(Q)$ defined by

$$Pre_{(u,d)}(W) = \{q \in Q \mid \exists u \in U \forall d \in D, \delta(q, u, d) \in W\}$$

and starting with $W = F$ determine the set of states for which Ursula can win the game if she chooses the right actions.

3. Specify a set valued feedback controller

$$g_u : Q \to P(U)$$

which ensures that Ursula will win the game if the initial state is one of these states. For the remaining states specify a set valued feedback controller

$$g_d : Q \to P(D)$$

that ensures that David will win the game.

For a general treatment of games of this type see P. Ramadge and M. Wonham, *The Control of Discrete Event Systems*, Proceedings of the IEEE, Vol. 77, No. 1, 1989, pp. 81-98.

6. Consider the timed automaton shown in Figure 5, and its region graph shown in Figure 6. Let $e_1 = (q_1, q_2)$ and $e_2 = (q_2, q_1)$. For the four regions, $P_1, \ldots, P_4$ shown in Figure 6 compute the predecessor operators Pre_{e_1} (states that can end up in P after transition e_1), Pre_{e_2} (states that can end up in P after transition e_2) and

$$Pre_C(P) = \{(q, x) \in Q \times \mathbb{R}^2 \mid \exists (q', x') \in P, t \geq 0, \text{ such that } (q = q') \wedge \left(x' = x + t \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\}$$

Show that all the predecessors are unions of partitions in the region graph (or the empty set). (Note: The domains are assumed to be the whole of X and so are omitted from the figure.)

7. A hybrid automaton, H, is called domain preserving if the state remains in the domain along all executions of the system. In other words, H is domain preserving if for all executions (τ, q, x) with $\tau = \{I_i\}_0^N$, for all $i = 1, \ldots, N$ and for all $t \in I_i$, $x_i(t) \in D(q_i(t))$ (N may be either finite or infinite).

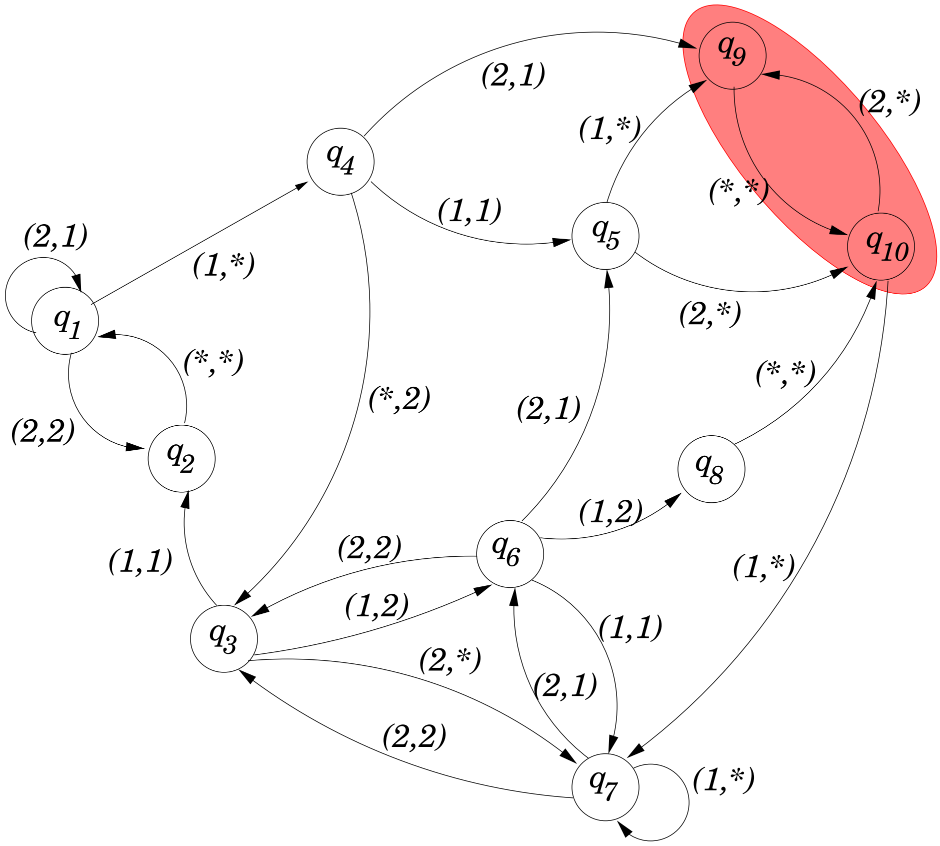

Figure 4: Game graph.

1. Show that H is domain preserving if and only if all reachable states $(\hat{q}, \hat{x})$ are such that $\hat{x} \in D(\hat{q})$. (Note: This is almost a tautology).
2. Assume that for all $\hat{q} \in Q$, $D(\hat{q})$ is a closed set. Show that H is domain preserving if
   - for all $(\hat{q}, \hat{x}) \in Init$, we have that $\hat{x} \in D(\hat{q})$; and,
   - for all $\hat{q} \in Q$, for all $\hat{x} \in D(\hat{q})$ and for all $(\hat{q}, \hat{q}') \in E$ such that $\hat{x} \in G(\hat{q}, \hat{q}')$, $R(\hat{q}, \hat{q}', \hat{x}) \subseteq D(\hat{q}')$.

   what if the system is blocking?

8. A thermostat system can be more accurately modelled by an impulse differential inclusion, $H_T = (X_T, F_T, R_T, J_T)$ with two state variables, $x = (x_1, x_2)$: the current room temperature x_1 and the steady state toward which the temperature is converging x_2 (which of course depends on whether the heater is on or off). Let $X_T = \mathbb{R}^2$, and

$$F_T(x_1, x_2) = ([a(x_1 - x_2), b(x_1 - x_2)], 0)$$

$$R_T(x_1, x_2) = \begin{cases} (x_1, 30 - x_2) \\ \quad \text{if } (x_1 \geq 21 \text{ and } x_2 \geq 20) \\ \quad \text{or } (x_1 \leq 19 \text{ and } x_2 \leq 20) \\ \emptyset \quad \text{otherwise} \end{cases}$$

$$J_T = \{x \in X_T \mid (x_1 \geq 22 \text{ and } x_2 \geq 20) \text{ or } (x_1 \leq 18 \text{ and } x_2 \leq 20)\},$$

with $a \leq b < 0$. Show that

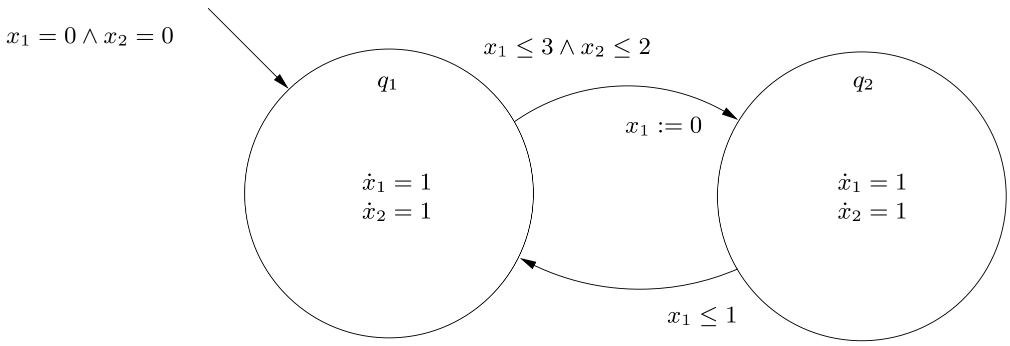

Figure 5: A timed automaton.

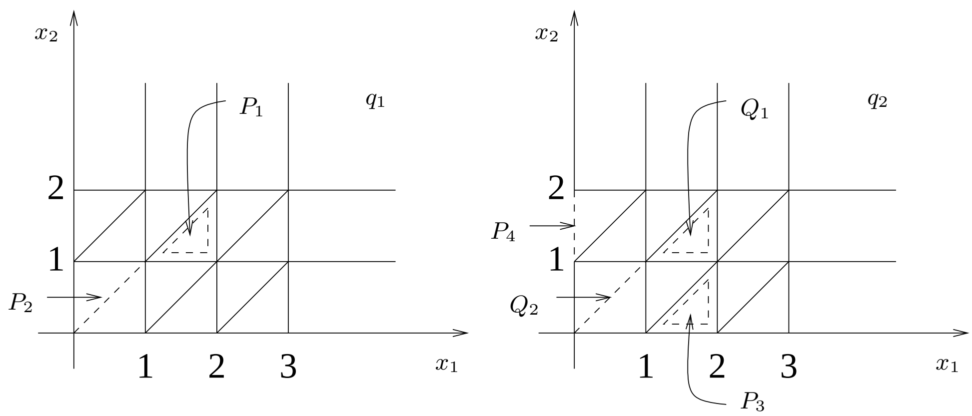

Figure 6: Region graph for the automaton of Figure 5

1. Infinite executions exist for all $x_0 \in X_T$.
2. The set $K = \{x \in X_T \mid x_2 \in \{0, 30\}\}$ is invariant.
3. The set $L = \{x \in X_T \mid x_1 \in [19, 21]\}$ is viable. Is L invariant?
4. The set $M = \{x \in X_T \mid x_1 \in [18, 22]\}$ is invariant.

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ENSIETA, February 2004