Computing reachable sets using a modified Hamilton-Jacobi equation

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Outline

I. Differential games

a) Dynamical systems with inputs and perturbations
b) Definition of the reachble set
c) Differential games

II. Hamilton-Jacobi equation

a) A level set formulation of reachability
b) Proof of the Hamilton-Jacobi formulation

III. Example

a) Pollution tax in civil engineering
b) 3D aircraft collision avoidance with application to ATC

Dynamical systems: inputs and perturbations

Dynamical System:

$$\frac{dx}{dt} = \dot{x} = f(x, a, b)$$

Player input, player I : $a(\cdot)$

Perturbation, player II: $b(\cdot)$

$$a(\cdot) \in \mathfrak{A}(t) \triangleq \{\phi : [t, 0] \to \mathcal{A} \mid \phi(\cdot) \text{ is measurable}\}$$

$$b(\cdot) \in \mathfrak{B}(t) \triangleq \{\phi : [t, 0] \to \mathcal{B} \mid \phi(\cdot) \text{ is measurable}\}$$

Dynamical systems: solution

Dynamical System:

$$\frac{dx}{dt} = \dot{x} = f(x, a, b)$$

Solution:

$$\xi_f(s; x, t, a(\cdot), b(\cdot)) : [t, 0] \rightarrow \mathbb{R}^n$$

where $\xi_f(t; x, t, a(\cdot), b(\cdot)) = x$

and $s \in [t, 0]$

Consequently, solutions satisfy:

$$\frac{d}{ds}\xi_f(s; x, t, a(\cdot), b(\cdot)) = f(\xi_f(s; x, t, a(\cdot), b(\cdot)), a(s), b(s))$$

Dynamical systems: nonanticipative strategies

Dynamical System:

$$\frac{dx}{dt} = \dot{x} = f(x, a, b)$$

Player input, player I : $a(\cdot)$

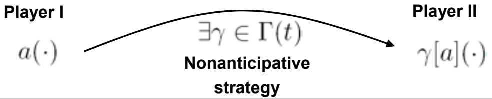
Perturbation, player II: $b(\cdot)$

Player II can only pick a nonanticipative strategy:

$$\gamma \in \Gamma(t) \triangleq \{\vartheta : \mathfrak{A}(t) \rightarrow \mathfrak{B}(t) \mid a(r) = \hat{a}(r) \text{ for almost every } r \in [t, s] \implies \vartheta[a](r) = \vartheta[\hat{a}](r) \text{ for almost every } r \in [t, s]\}.$$

Nonanticipative strategies (cont.)

$$\gamma \in \Gamma(t) \triangleq \{\vartheta : \mathfrak{A}(t) \rightarrow \mathfrak{B}(t) \mid a(r) = \hat{a}(r) \text{ for almost every } r \in [t, s] \implies \vartheta[a](r) = \vartheta[\hat{a}](r) \text{ for almost every } r \in [t, s]\}.$$



Player I

$a(\cdot)$

$\exists \gamma \in \Gamma(t)$

Nonanticipative strategy

Player II

$\gamma[a](\cdot)$

Dynamical systems: nonanticipative strategies

Dynamical System:

$$\frac{dx}{dt} = \dot{x} = f(x, a, b)$$

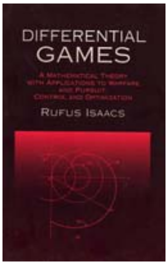
Player I chooses to play $a(\cdot)$

Based on the choice of player I, player II plays $\gamma[a](\cdot)$

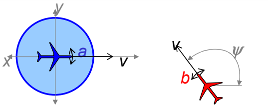
As a result, a generic trajectory $\xi_f(s; x, t, a(\cdot), b(\cdot))$

becomes $\xi_f(s; x, t, a(\cdot), \gamma[a](\cdot))$

Conflicting goals: differential games, 1957



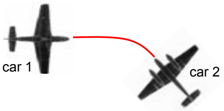
$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \psi \end{bmatrix} = \begin{bmatrix} -v_a + v_b \cos\psi + \omega_a y \\ v_b \sin\psi - \omega_a x \\ \omega_b - \omega_a \end{bmatrix}$$



‘evader’
(control)

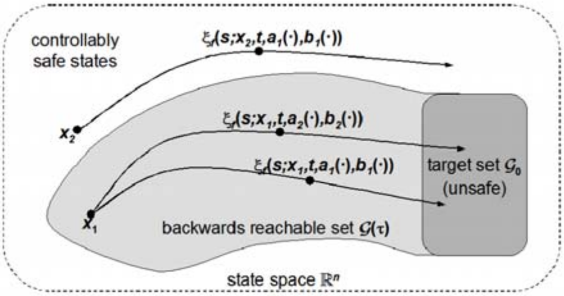
‘pursuer’
(disturbance)

The game of two cars…
The homicidal chauffeur…

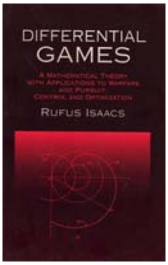


Conflicting goals: differential games

Player II wants to push player I into an unsafe set:



Conflicting goals: differential games, 1957



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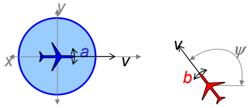

‘evader’
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The game of two cars…
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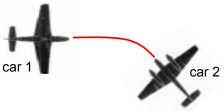
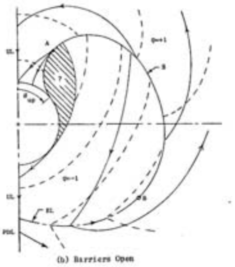


Seminal Isaacs approach: retrogrde path eqns.

THE HOMICIDAL CHAUFFEUR - A DIFFERENTIAL GAME

A DISSERTATION
SUBMITTED TO THE DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS
AND THE COMMITTEE ON GRADUATE STUDIES
OF STANFORD UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

By
Antony Willits Merz
March 1971



(b) Barriers Open

FIGURE 2.5. Trajectories and Isochrones in the Classical Game

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II. Hamilton-Jacobi equation

- a) A level set formulation of reachability
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III. Example

- a) Pollution tax in civil engineering
- b) 3D aircraft collision avoidance with application to ATC

Reachable set: mathematical definition

Target set:

$$\mathcal{G}_0 = \{x \in \mathbb{R}^n \mid g(x) \leq 0\}$$

Reachable set:

$$\mathcal{G}(\tau) \triangleq \{x \in \mathbb{R}^n \mid \exists \gamma \in \Gamma(t), \forall a(\cdot) \in \mathfrak{A}(t), \exists s \in [t, 0], \\ \xi_f(s; x, t, a(\cdot), \gamma[a](\cdot)) \in \mathcal{G}_0\}.$$

$$\tau = -t$$

Main result: reachability theorem

Theorem: Let $v : \mathbb{R}^n \times [-T, 0] \to \mathbb{R}$ be the viscosity solution of the terminal value HJI PDE

$$D_t v(x,t) + \min[0, H(x, D_x v(x,t))] = 0, \\ v(x, 0) = g(x),$$

where

$$H(x,p) = \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} p^T f(x, a, b).$$

Then the zero sublevel set of v describes $\mathcal{G}(\tau)$

$$\mathcal{G}(\tau) = \{x \in \mathbb{R}^n \mid v(x,t) \leq 0\}.$$

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Why do we want to solve a HJE PDE?

I. Retrograde path equations are tedious and only work in specific cases

II. The viscosity solution of the HJE PDE provides the unique correct solution to the control problem

III. Efficient numerical techniques have been designed for solving the HJE PDE accurately

IV. The viscosity solution is smooth, unlike for other equivalent techniques to solve the same problem

V. The viscosity of the HJE PDE provides additional useful information for the differential game problem

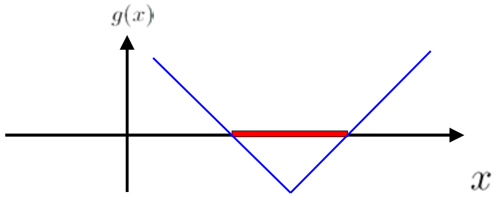
---

What is the viscosity solution of the HJE PDE?

We start with a terminal condition:

$v(x,0) = g(x)$

Let us assume that the terminal condition represents the signed Distance of a given point to the target set:



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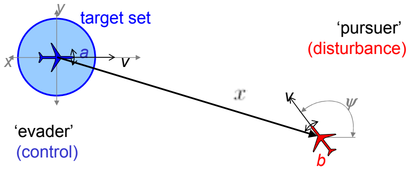

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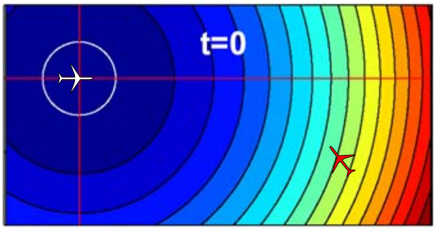

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What is the viscosity solution of the HJE PDE?

We start with a terminal condition:

$v(x,0) = g(x)$

Integrate the HJE PDE

$$D_t v(x,t) + \min[0, H(x, D_x v(x,t))] = 0$$

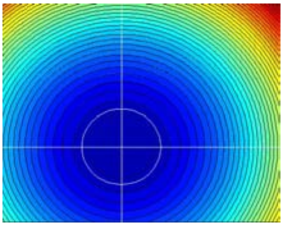
$v(x,t)$

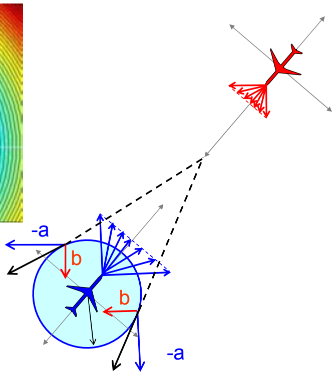
Find the solution at time t

The subzero level set of the function v **describes the reachable set**

$$\mathcal{G}(\tau) = \{x \in \mathbb{R}^n \mid v(x,t) \leq 0\}$$

Example: one chance collision (Isaacs 1957)





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Proof: outline

I. Start with a known theorem of calculus of variations

II. Take into account that the trajectories can enter the target set and exit

III. Modify the input of the disturbance accordingly

IV. Compute the corresponding HJE PDE

Step I: start from calculus of variations

Consider solving:

$$v(x,t) = \inf_{\tilde{\gamma} \in \tilde{\Gamma}(t)} \sup_{a(\cdot) \in \mathfrak{A}(t)} g(\xi_{\tilde{f}}(0; x, t, a(\cdot), \tilde{\gamma}[a](\cdot)))$$

(for now ignore the tildas)

Step I: start from calculus of variations

Consider solving:

$$v(x,t) = \inf_{\tilde{\gamma} \in \tilde{\Gamma}(t)} \sup_{a(\cdot) \in \mathfrak{A}(t)} g(\xi_{\tilde{f}}(0; x, t, a(\cdot), \tilde{\gamma}[a](\cdot)))$$

(for now ignore the tildas)

Theorem: The value function $v(x,t)$ of our game is the viscosity solution of the Hamilton-Jacobi-Isaacs terminal value PDE

$$\begin{aligned} D_t v(x,t) + \tilde{H}(x, D_x v(x,t)) &= 0, \\ v(x,0) &= g(x), \end{aligned}$$

where

$$\tilde{H}(x,p) = \max_{a \in \mathcal{A}} \min_{\tilde{b} \in \tilde{\mathcal{B}}} p^T \tilde{f}(x, a, \tilde{b}).$$

Proof: outline

I. Start with a known theorem of calculus of variations

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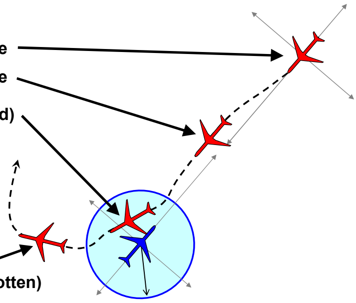
III. Modify the input of the disturbance accordingly

IV. Compute the corresponding HJE PDE

Step II: trajectories can enter and leave

Cost of the game: $\inf_{\tilde{\gamma}\in\tilde{\Gamma}(t)} \sup_{a(\cdot)\in\mathfrak{A}(t)} g(\xi_{\tilde{f}}(0;x,t,a(\cdot),\tilde{\gamma}[a](\cdot)))$



Positive

Positive

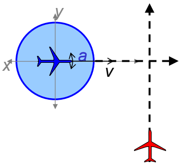
Negative (conflict detected)

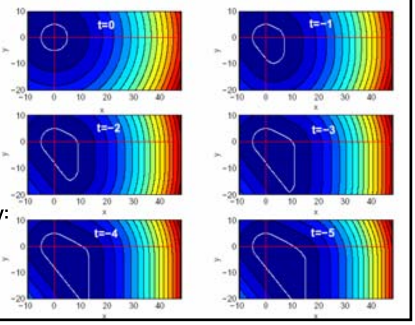
Positive again (conflict forgotten)

Step II: easy fix (but wrong)

Compute the minimum of this quantity for time Range of interest.

$\inf_{\tilde{\gamma}\in\tilde{\Gamma}(t)} \sup_{a(\cdot)\in\mathfrak{A}(t)} g(\xi_{\tilde{f}}(0;x,t,a(\cdot),\tilde{\gamma}[a](\cdot)))$





Nonanticipative strategy: mimic the blue aircraft

Proof: outline

I. Start with a known theorem of calculus of variations

II. Take into account that the trajectories can enter the target set and exit

III. Modify the input of the disturbance accordingly

IV. Compute the corresponding HJE PDE

Step III: real fix (freezing input)

Introduce the "freezing input"

$$\underline{b}(\cdot) \in \mathfrak{B}(t) \triangleq \{\phi : [t,0] \to [0,1] \mid \phi(\cdot) \text{ is measurable}\}$$

Consider the augmented input of player II

$$\tilde{b} = [b \quad \underline{b}] \in \mathcal{B} \times [0,1]$$

Consider the augmented dynamicsof the two players

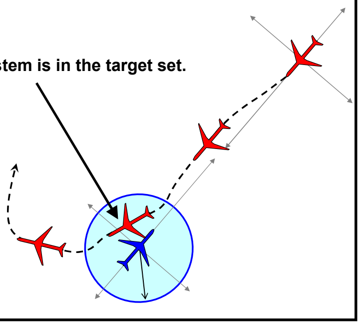
$$\tilde{f}(x,a,\tilde{b}) \triangleq \underline{b} f(x,a,b)$$

Consider the trajectories of this augmented system

$$\xi_{\tilde{f}}(s; x, t, a(\cdot), \tilde{b}(\cdot))$$

Player II has the possibility to "freeze" the system...

Step III: trajectories can enter and leave



... while the state of the system is in the target set.

Proof: outline

I. Start with a known theorem of calculus of variations

II. Take into account that the trajectories can enter the target set and exit

III. Modify the input of the disturbance accordingly

IV. Compute the corresponding HJE PDE

Step IV: compute the corresponding HJE PDE

The trajectories of the original system and of the augmented system are equivalent:

$$\xi_f(\sigma(s); x, t, a(\sigma^\dagger(\cdot)), b(\sigma^\dagger(\cdot))) = \xi_{\tilde{f}}(s; x, t, a(\cdot), \tilde{b}(\cdot))$$

where $\sigma^\dagger$ **is the pseudo inverse of the function**

$$\sigma(s) \triangleq t + \int_t^s \underline{b}(\lambda)\, d\lambda$$

Step IV: compute the “freezing” Hamiltonian

Start with the Hamiltonian for the system with “freezing input”

$$\begin{aligned}
\tilde{H}(x,p) &= \max_{a \in \mathcal{A}} \min_{\tilde{b} \in \tilde{\mathcal{B}}} p^T \tilde{f}(x, a, \tilde{b}), \\
&= \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} \min_{\underline{b} \in [0,1]} p^T (\underline{b} f(x, a, b)), \\
&= \min_{\underline{b} \in [0,1]} \underline{b} \left(\max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} p^T f(x, a, b) \right) \\
&= \min[0, H(x,p)].
\end{aligned}$$

Final result in terms of the original dynamics only.

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a) Pollution tax in civil engineering
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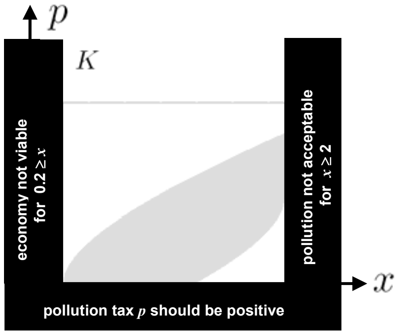
Example (civil engineering): pollution-tax

$$\begin{cases} x' = ax - p \\ p' = u \end{cases}$$

$$u \in [\dot{p}_{\min}, \dot{p}_{\max}]$$



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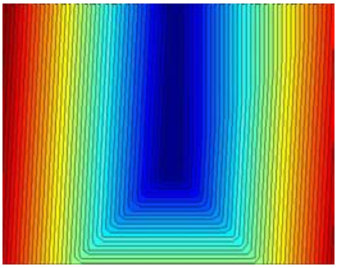

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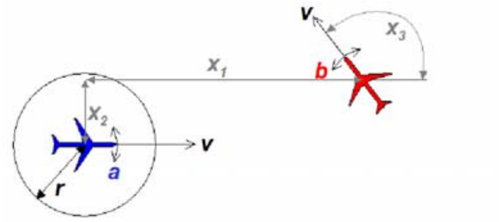
3D example: collision avoidance



$$\dot{x} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -v_a + v_b \cos x_3 + a x_2 \\ v_b \sin x_3 - a x_1 \\ b - a \end{bmatrix} = f(x, a, b)$$

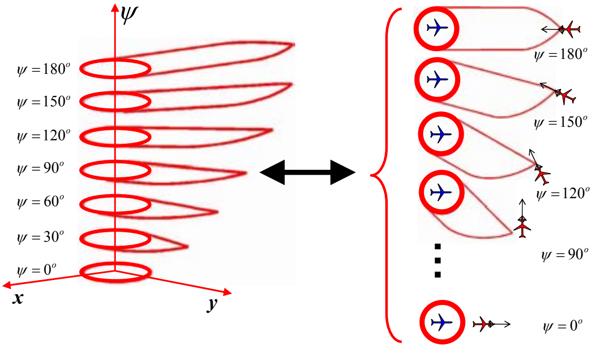
3D example: collision avoidance

$$\begin{aligned} H(x,p) &= \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} \left[p^T f(x,a,b) \right], \\ &= \left(\begin{aligned} &- p_1 v_a + p_1 v_b \cos x_3 + p_2 v_b \sin x_3 \\ &\qquad + \alpha |p_1 x_2 - p_2 x_1 - p_3| - \beta |p_3| \end{aligned} \right) \end{aligned}$$

$$\begin{aligned} \mathcal{A} &= [-\alpha, +\alpha] \\ \mathcal{B} &= [-\beta, +\beta] \end{aligned}$$

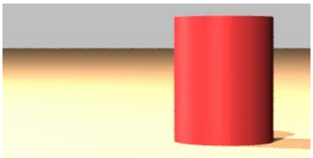
$$\begin{aligned} \mathcal{G}_0 &= \{ x \in R^3 | x_1^2 + x_2^2 \leq r^2 \}, \\ g(x) &= \sqrt{x_1^2 + x_2^2} - r, \end{aligned}$$

3D example: collision avoidance

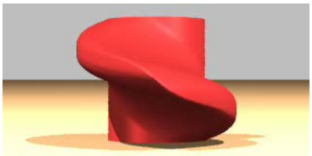


Computation of the reachable set

Growth of the reachable set

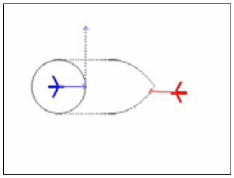


3D view of the reachable set

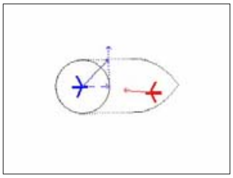


Danger zone - unsafe set (reachable set)

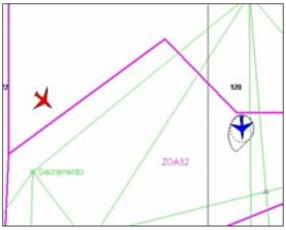
Intruder aircraft starts outside the reachable set

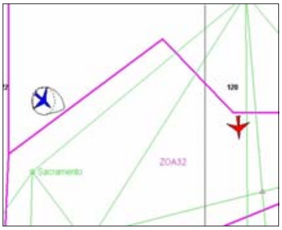


Intruder aircraft starts inside the reachable set



Applications to ETMS data





An algorithm implemented as embedded software, which runs onboard each vehicle, and can:

1) Detect conflicts
2) Resolve the conflicts

Next lecture...