Alternate ways to compute the reachable sets; applications from viability theory

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Outline

I. Viability theory
- a) The viability kernel and the capture basin
- b) Tangential conditions
- c) Interpretation of the barrier

II. Example: the Zermelo swimmer

III. Minimum time to reach
- a) The epigraph of the value function
- b) Zermelo swimming through obstacles
- c) Back to the Hamilton-Jacobi equation

IV. Examples
- a) Exit time in a labyrinth,
- b) Fractals, Lorenz's attractor
- c) Landing envelopes
- d) Zermelo takes the metro

Set valued dynamics

$$x'(t) = f(x(t), u(t)) \quad u(t) \in U \tag{0.1}$$

Dynamical system
Input varies in a given set

$$x'(t) \in F(x(t)) := \{f(x(t), u \mid u \in U\} \tag{0.2}$$

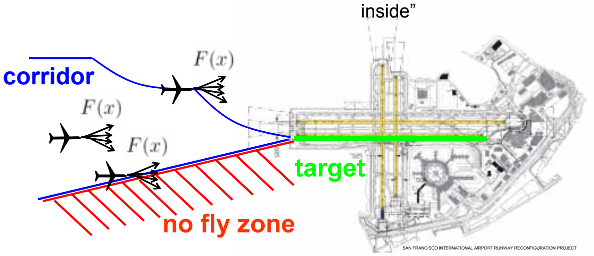
Set valued dynamics: takes all inputs into account
Resulting dynamics: set valued dynamics

$x(\cdot)$ Generic trajectory of the system

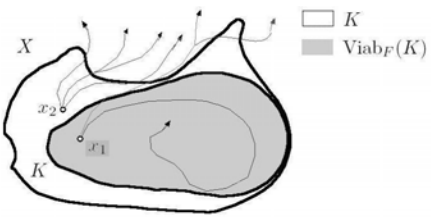
Interpretation of the set valued dynamics

Set valued dynamics $F(x)$

"Close to the boundary of the set such that the aircraft wants to stay, it should have a required input to make it stay inside"

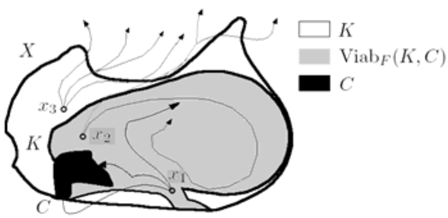


The viability kernel



$$\text{Viab}_F(K) := \{x_0 \in K |\ \exists x(\cdot)\ solving\ (0.2),\ s.t.\ x(0) = x_0,\ \forall t > 0,\ x(t) \in K\}$$

The viability kernel with target



$$\text{Viab}_F(K, C) := \{x_0 | x_0 \in K,\ \exists x(\cdot)\ solving\ (0.2),\ x(0) = x_0\ s.t.\ \forall t > 0,\ x(t) \in K,\ or\ \exists t > 0,\ x(t) \in C,\ and\ \forall s \in [0, t],\ x(s) \in K\}$$

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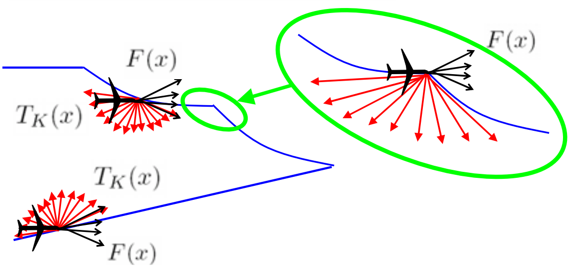
The contingent cone

$$T_K(x) \; := \; \text{Limsup}_{h \to 0+} \frac{K - x}{h}$$

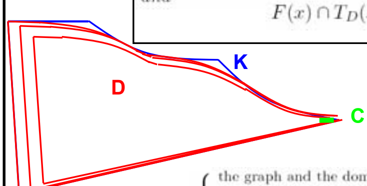


$$\text{Limsup}_{n \to \infty} K_n \; := \; \left\{ x \in E \;|\; \liminf_{n \to \infty} d(x, K_n) = 0 \right\}$$

Characterization of the viability kernel

Theorem
Let us assume that F is Marchaud and that the target $C \subset K$ and K are closed. The viability kernel $\text{Viab}_F(K, C)$ of the subset K with target C is the largest closed subset D satisfying $C \subset D \subset K$ and

$$F(x) \cap T_D(x) \neq \emptyset$$



F is a Marchaud map if

$$\begin{cases} \text{the graph and the domain of } F \text{ are nonempty and closed} \\ \text{the values } F(x) \text{ of } F \text{ are convex} \\ \text{the growth of F is linear:} \\ \exists\, c > 0 \;|\; \forall x \in X, \;\; \|F(x)\| := \sup_{v \in F(x)} \|v\| \;\leq\; c(\|x\| + 1) \end{cases}$$

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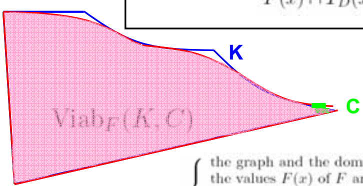

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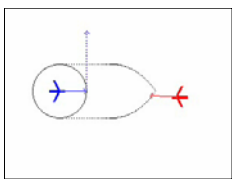
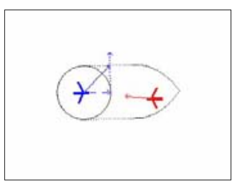
Interpretation of the tangential conditions

Intruder aircraft starts outside the reachable set



Intruder aircraft starts inside the reachable set



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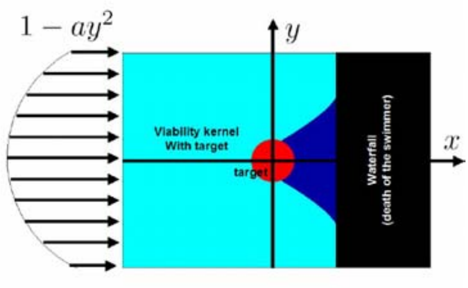
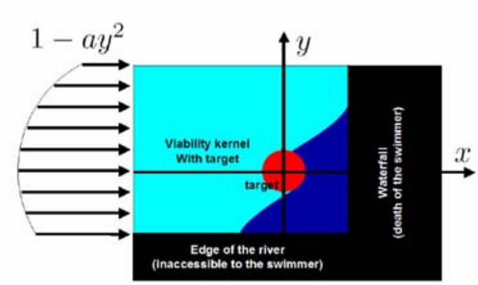
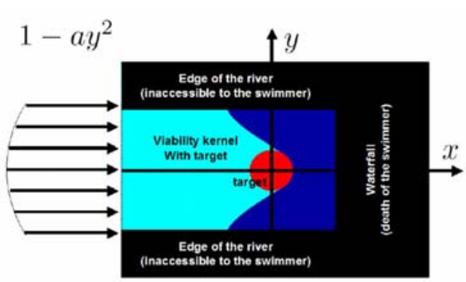
The Zermelo swimmer (I)



The Zermelo swimmer (II)



The Zermelo swimmer (III)



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Epigraph of the minimum time to reach function

y

Example:
One dimensional target C
Set valued dynamics $F(x)$

How to compute the minimum time to reach C ?
Add one dimension y **for time:**

C x

$$\dot{x}(t) \in F(x(t))$$

Epigraph of the minimum time to reach function

y

Dynamics along the horizontal axis

"count down"

$$\Phi(x) = \begin{cases} F(x) \times \{-1\} & \text{if } x \notin C \\ \overline{\text{co}}\{F(x) \times \{-1\}, \{0,0\}\} & \text{if } x \in C \end{cases}$$

Augment dynamics along the y **axis**

C x

$$\dot{x}(t) \in F(x(t))$$

Epigraph of the minimum time to reach function

$$\Phi(x) = \begin{cases} F(x)\times\{-1\} & \text{if } x\notin C \\ \overline{\text{co}}\{F(x)\times\{-1\},\{0,0\}\} & \text{if } x\in C\end{cases}$$

convex hull — of the dynamics — with zero

Augment dynamics along the y axis
So that it is possible to stop in the target

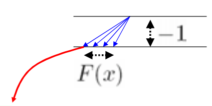
y, x, C

$\dot{x}(t) \in F(x(t))$

Epigraph of the minimum time to reach function

$$\Phi(x) = \begin{cases} F(x)\times\{-1\} & \text{if } x\notin C \\ \overline{\text{co}}\{F(x)\times\{-1\},\{0,0\}\} & \text{if } x\in C\end{cases}$$

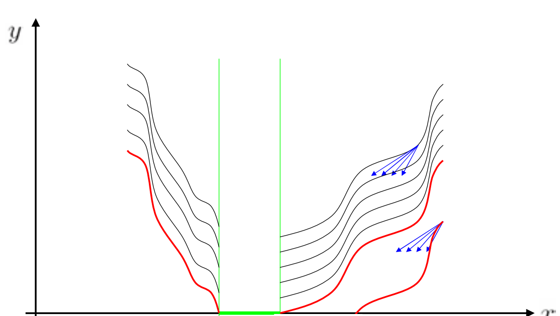
convex hull — of the dynamics — with zero



y, x, C

$\dot{x}(t) \in F(x(t))$

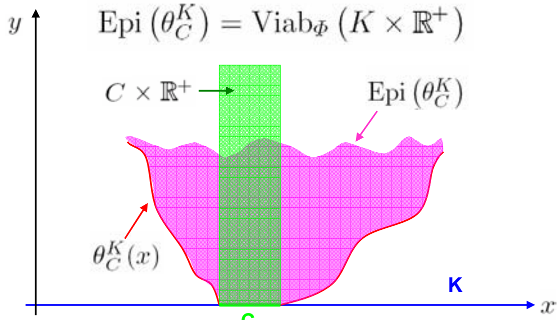
Epigraph of the minimum time to reach function



$\dot{x}(t) \in F(x(t))$

Epigraph of the minimum time to reach function

$$\mathrm{Epi}\left(\theta_C^K\right) = \mathrm{Viab}_\Phi\left(K \times \mathbb{R}^+\right)$$



$$\mathrm{Epi}\left(\theta_C^K\right) := \{(x,y) \in K \times \mathbb{R}^+ : y \geq \theta_C^K(x)\}$$

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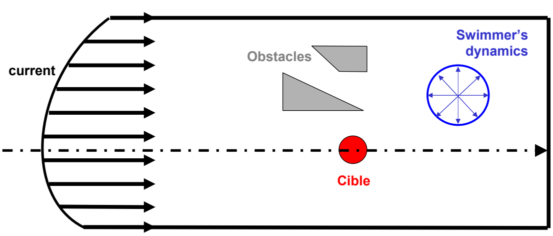
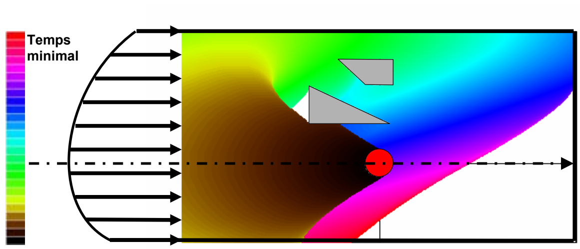
Example: Zermelo swimmer with obstacles

River with nonuniform current and obstacles



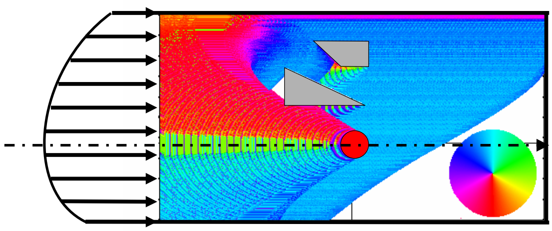
Minimal time

Epigraph of the minimal time to reach is the capture basin of the augmented target with augmented dynamics



Optimal control

Optimal control for this problem is the optimal swim direction



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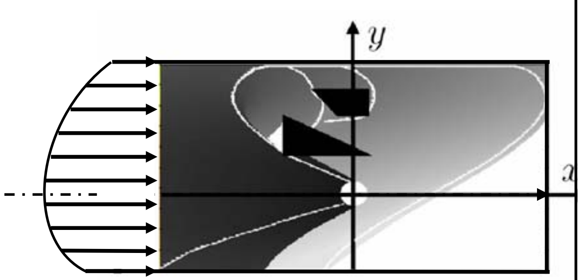
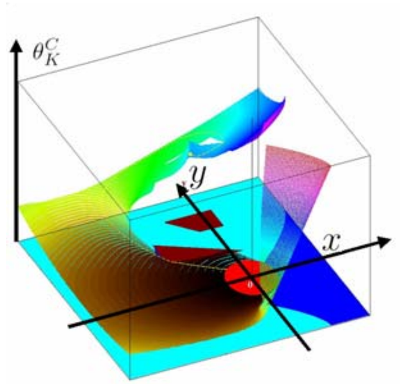


Graph of the minimum time to reach function



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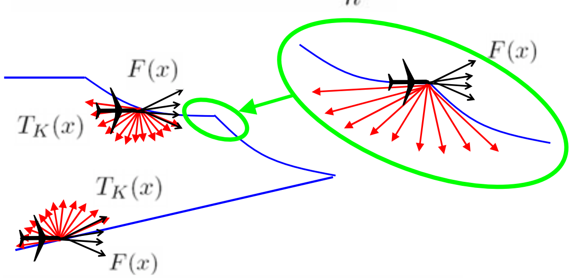
Link with the Hamilton-Jacobi equation

$$T_K(x) := \mathrm{Limsup}_{h \to 0+} \frac{K - x}{h}$$



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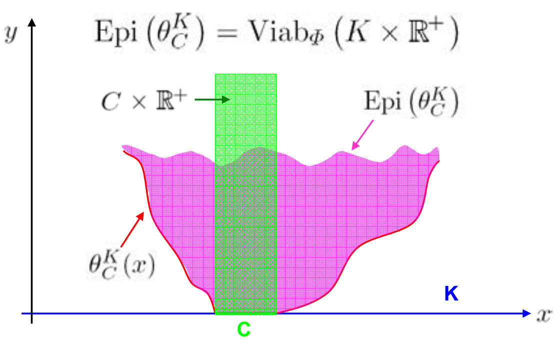
Epigraph of the minimum time to reach function



$$\forall (x, y) \in Epi(\vartheta_C^K), \qquad \Phi(x, y) \cap T_{Epi(\vartheta_C^K)}(x, y) \neq \emptyset.$$

Expressing these conditions leads to the static HJE

The value function satisfies the static HJE

$$\forall\, x \in Dom(V), \qquad H\left(x, -\frac{d}{dx}V(x)\right) = 0$$

With the following Hamiltonian

$$H(x, p) := \max_{u \in U} \langle f(x, u), p \rangle - 1$$

Link with the Hamiltoon-Jacobi equation

$$\Phi(x) = \begin{cases} F(x) \times \{-1\} & if \;\; x \notin C \\ \overline{\mathrm{co}}\{F(x) \times \{-1\}, \{0, 0\}\} & if \;\; x \in C \end{cases}$$

$$\forall (x, y) \in Epi(\vartheta_C^K), \qquad \Phi(x, y) \cap T_{Epi(\vartheta_C^K)}(x, y) \neq \emptyset.$$

$$\forall\, (x, y) \in Epi(\vartheta_C^K), \qquad \Phi(x, y) \cap Epi(D_\uparrow \vartheta_C^K)(x)) \neq \emptyset$$

Link with the static Hamilton-Jacobi equation

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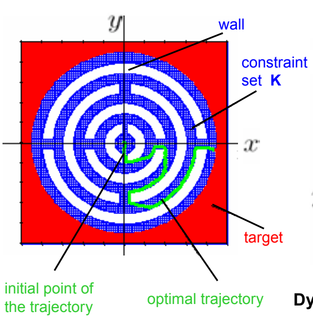
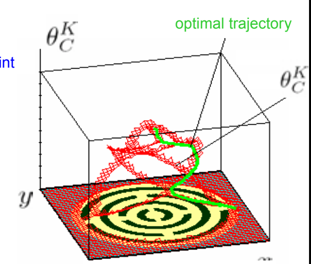
Example of minimum time to reach





Dynamics: $\begin{cases} x'(t) = \gamma \cos\theta \\ y'(t) = \gamma \sin\theta \end{cases}$

$$\gamma \in [0, v_{\max}], \;\; \theta \in [0, 2\pi]$$

Fractals: the Mandelbrot function

$$\begin{cases} x_{j+1} = \varphi(x_j, u) \\ x_0 = x \end{cases} \qquad \mu(x,u) := \sup_{j \geq 0} \|x_j\|$$

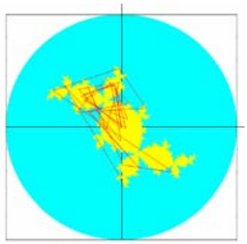
Discrete dynamical system

Constant input u

Initial condition x

$$\varphi(x_j) = x_j^2 + u$$

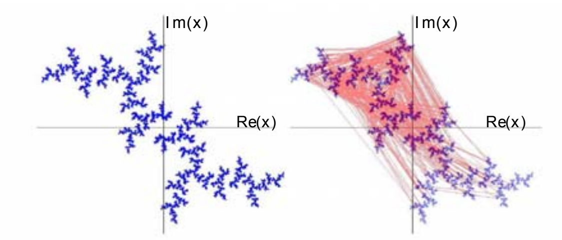
Which x are such that after an infinite number of iterations, is x_j still in the ball



$$\mu(x,y) \leq y \textit{ if and only if } x \in \mathrm{Viab}_{\varphi(\cdot,u)}(B(0,y))$$

Mandelbrot function for an other value of u

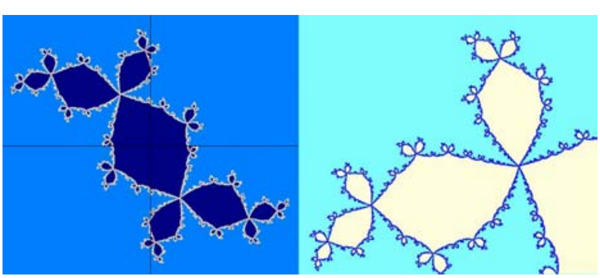
$$\begin{cases} x_{j+1} = \varphi(x_j, u) \\ x_0 = x \end{cases} \qquad \mu(x,u) := \sup_{j \geq 0} \|x_j\|$$



$$\mu(x,y) \leq y \textit{ if and only if } x \in \mathrm{Viab}_{\varphi(\cdot,u)}(B(0,y))$$

Fragility of the viability kernel

$$\begin{cases} x_{j+1} = \varphi(x_j, u) \\ x_0 = x \end{cases} \qquad \mu(x,u) := \sup_{j \geq 0} \|x_j\|$$



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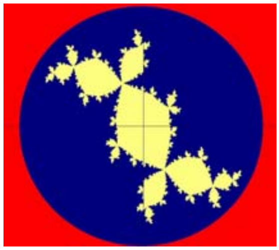
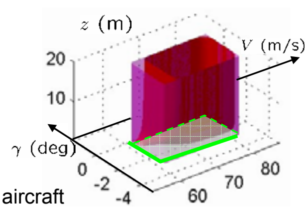


Which x are such that after an infinite number of iterations, is x_j still in the ball

Illustration of the capture basin



Landing envelope of a DC9-30 aircraft

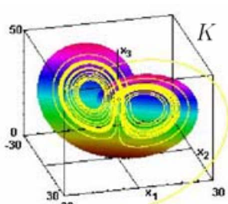
Constraint set: flight envelope

Target set: set admissible touch down parameters

Landing envelope: set of flight parameters from which a safe touch down is possible is the capture basin

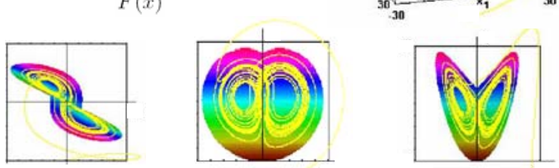
The Lorenz attractor

$$\begin{cases} \dot{x}(t) = \sigma y(t) - \sigma x(t) \\ \dot{y}(t) = r x(t) - y(t) - x(t)z(t) \\ \dot{z}(t) = x(t)y(t) - bz(t) \end{cases}$$

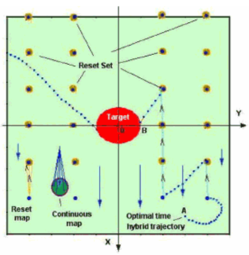


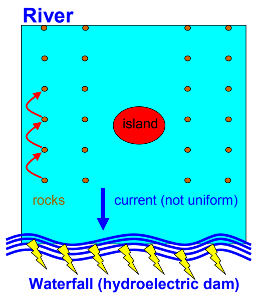
K

$F(x)$

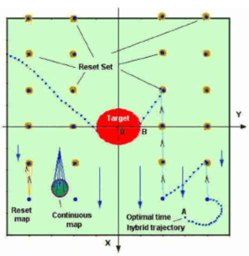


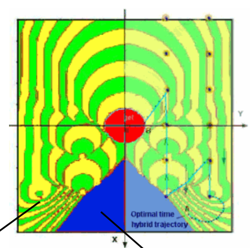
Example: the Zermelo problem





Example: the Zermelo problem with multiple islands





In this region, the swimmer can:

1. **swim / jump / swim / jump … forever**
2. **swim / jump /swim … reach the island**

Here, the swmimmer goes into the waterfall (and dies)

Hybrid system: walk or take the metro?