II. THE BALL AND BEAM EXAMPLE

Consider the familiar ball and beam experiment, found in many undergraduate control laboratories, depicted in Fig. 1. The beam is made to rotate in a vertical plane by applying a torque at the center of rotation and the ball is free to roll (with one degree of freedom) along the beam. We require that the ball remain in contact with the beam and that the rolling occur without slipping, which imposes a constraint on the rotational acceleration of the beam. Our goal is to track any trajectory from a class of admissible trajectories.

Let the moment of inertia of the beam be $J$, the mass and moment of inertia of the ball be $M$ and $J_r$, respectively, the radius of the ball be $R$, and the acceleration of gravity be $g$. Choosing the beam angle $\theta$ and the ball position $r$ as generalized position coordinates for the system, the Lagrangian equations of motion are given by

$$0 = \frac{J_0}{R^2} \dot{\theta} + MG \sin \theta - Mr^2 \ddot{\theta}$$
$$\tau = (Mr^2 + J + J_r) \ddot{\theta} + 2Mr \dot{\theta} + MGr \cos \theta$$

where $\tau$ is the torque applied to the beam. Defining $B := M / (J_r / R^2 + M)$ and changing the coordinates in the input space using the invertible nonlinear transformation

$$\tau = 2Mr \dot{\theta} + MGr \cos \theta + (Mr^2 + J + J_r) u$$

to define a new input $u$ the system can be written in state-space form as

$$\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
x_3 \\
x_3 \\
0
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
\frac{1}{2}
\end{bmatrix} u$$

$$y = \begin{bmatrix} x_1 \end{bmatrix}$$

where $x = (x_1, x_2, x_3, x_4)^T := (r, \ddot{r}, \theta, \dot{\theta})^T$ and $y = h(x) := r$.

The system output $y(t)$ is to track a desired trajectory $y_d(t)$, i.e., $y(t) \rightarrow y_d(t)$ as $t \rightarrow \infty$. Following the usual procedure [5], we differentiate the output until the input appears

$$\ddot{y} = Bx_1x_2 - BG \sin x_3,$$

$$y^{(3)} = Bx_1x_2 - BGx_4 \cos x_3 + 2Bx_1x_3 u.$$
The integrability condition which requires the involutivity of
\[ \text{span} \{ g, \text{ad}_g f, \cdots, \text{ad}_g^{n-2} f \} \] (2.5)
since \( \{ g, \text{ad}_g f \} = (2Bx_1 - 2Rx_2, 0, 0)^T \) does not lie in (2.5). Here \( \text{ad}_g f \) denotes the iterated Lie bracket \( [f, \cdots, [f, g] \cdots] \). Thus, it is not possible to fully linearize the ball and beam system.

III. APPROXIMATE INPUT-OUTPUT LINEARIZATION: THE BALL AND BEAM SYSTEM

In this section, we show that, by appropriate choice of vector fields close to the system vector fields, we can design a feedback control law to achieve approximate output tracking. The control law will, in fact, be the exact output tracking control law for an approximate system. We construct two input-output linearizable approximations for the ball and beam system. In each case, we construct a nonlinear change of coordinates \( \xi = \Phi(x) \), and a state feedback \( u(x, v) = \alpha(x) + \beta(x) v \), to make the system look like a chain of integrators perturbed by higher order terms \( \dot{\psi}(x, v) \) as depicted in Fig. 2. Each successive component of the coordinate change is constructed by differentiating the previous one along the system trajectories and discarding some higher order terms. (The first component is chosen to be the system output.) The performance of these designs is compared to the performance of a linear controller based on the standard Jacobian approximation.

Approximate tracking is achieved for each design by choosing \( u \) to equate \( v = \dot{b}(x) + a(x) u \) with

\[
\begin{align*}
  v &= y_i^B(t) + \alpha_s(y_i^B(t) - \phi_s(x)) + \alpha_d(y_i^d(t) - \phi_d(x)) \\
  &\quad + \alpha_s(y_i^d(t) - \phi_s(x)) + \alpha_0(y_i^d(t) - \phi_i(x))
\end{align*}
\] (3.1)

making the error system into an exponentially stable linear system perturbed by small nonlinear terms. Here, the \( \alpha_s \) are chosen so that \( s^4 + \alpha_s s^3 + \alpha_s s^2 + \alpha_s s + \alpha_0 \) is a Hurwitz polynomial.

For each approximation, we present simulation results depicting the output error \( y_i^d(t) - \phi_i(x(t)) \), the angle of the beam \( \theta(t) = x_3(t) \), and the neglected nonlinearity \( \dot{x}_2 \), and the input torque \( \tau(t) \), for three desired trajectories given by \( y_i^d(t) = A \cos \pi t / 5 \), with \( A = 1, 2, \) and \( 3 \) and initial conditions for \( (r, \theta) \) of (1, 0.0564), (2, 0.1129), and (3, 0.1698), respectively. The system parameters used for simulation are \( M = 0.05 \) kg, \( R = 0.01 \) m, \( J = 0.02 \) kg m\(^2\), \( J_s = 2 \times 10^{-6} \) kg m\(^2\), and \( G = 9.81 \) m/s\(^2\) (thus \( B = 0.7143 \)). All closed-loop poles are placed at \(-2\). Note that the desired trajectories require much greater deviations than possible with most experimental setups.

Approximation 1 (Modification of \( f \))

Since the system fails to have a well-defined relative degree because of the centrifugal acceleration term \( Bx_1^2 x_3^2 \), we design our first approximate system by simply neglecting the term \( Bx_1^2 x_3^2 \). Let \( \xi_1 = \phi_i(x) = h(x) \). Then, choosing \( \phi_i(x) \) at each step, we have

\[
\begin{align*}
  \dot{\xi}_1 &= \xi_2 \\
  \dot{\xi}_2 &= -B \xi_1 \sin \xi_3 + B \xi_1 \xi_3^2 \\
  \dot{\xi}_3 &= -B \xi_1 \cos \xi_3 \\
  \dot{\xi}_4 &= b(x) + a(x) u
\end{align*}
\] (3.2)

As expected, by neglecting the centrifugal term (which is higher order), we obtain an approximate system with a well-defined relative degree. Note that the choice of what to neglect (i.e., \( \psi_i(x) \)) leads to a specification of the coordinate transformation \( \Phi(x) \). In this case, the approximate system has been obtained by a simple modification of the \( f \) vector field (i.e., by neglecting \( \psi_i(x) \)).

The simulation results in Fig. 3 show that the closed-loop system provides good tracking. The tracking error increases in a nonlinear fashion as the amplitude \( A \) of the desired trajectory is increased. This is expected since the neglected term \( \psi_i(x) \) is a nonlinear function of the state. A good \( a \) priori estimate of the mismatch of the approximate system for a desired trajectory can be calculated using

\[ \dot{\psi}(\Phi^{-1}(x, \tau, y_{i,1}^d, y_{i,2}^d)) \]

where \( \Phi^{-1}: x \rightarrow x \) is the inverse of the coordinate transformation. This in turn may be a useful way to define a class of trajectories that the system can track with small error.

Approximation 2 (Modification of \( g \))

For this approximation, we only discard terms that we must in order to obtain an approximate system with a well-defined relative degree, that is, we modify the \( g \) vector field. Again, let \( \xi_1 = \phi_i(x) \).
\[ \dot{x}_i = \frac{x_2}{x_1} - \frac{\phi(x)}{x_2} \]
\[ \dot{\xi}_1 = -BG \sin x_1 + Bx_1 x_2^2 \]
\[ \dot{\xi}_2 = -BGx_4 \cos x_2 + Bx_2 x_4^2 + 2Bx_1 x_4 u \]
\[ \dot{\xi}_3 = \frac{\xi_4 - \phi(x)}{x_3} \]
\[ \dot{\xi}_4 = B^2 x_1 x_4^2 + B(1 - B) x_4^2 \sin x_3 + \frac{(-BG \cos x_3 + 2Bx_1 x_4) u}{(x_3) u} \]

In order to guarantee a well-defined relative degree, we were forced to discard \( \phi_3(x, u) = 2Bx_1 x_4 u \) since \( x_1, x_4 \) is zero at \( x = 0 \) but not identically zero in a neighborhood of \( x = 0 \).

The advantage of writing the system in mixed \( x \) and \( \xi \) coordinates as in (3.3) is that it is easy to identify the terms in the \( g \) vector field that should be dropped (in this case, \( \phi_3 \)). However, when this modification is expressed completely in \( x \) (or \( \xi \)) coordinates it is, in fact, very complicated. For example, in \( x \) coordinates, dropping \( \phi_3 \) corresponds to subtracting

\[ Dg(x) = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right] \]

and

\[ \psi_3(x, u)/u \]

The Jacobian approximation is, of course, obtained by replacing the \( f \) vector field by its linear approximation and the \( g \) vector field by its constant approximation, that is, by neglecting \( \psi_2 \) and \( \phi_3 \). Fig. 5 shows the simulation results from the Jacobian approximation, indicating that the control system is no longer stable for \( A = 3 \).

Table I provides a direct comparison of the maximum error magnitude \( |e| = |x_e - x_i| \) between 30 and 40 s for the three approximations. Note that Approximation 2 provides somewhat better tracking for this class of inputs than Approximation 1. It is of practical interest that Approximations 1 and 2 continue to approximate the ball and beam system for trajectories with \( A \) more than two times as large as the value at which the Jacobian approximation resulted in an unstable system. Initial conditions for the simulation with \( A = 6 \) are \( (r, \theta) = (6, 0.345) \).

IV. ANALYSIS OF APPROXIMATE LINEARIZATION

In this section, we will consider single-input single-output systems of the form

\[ \dot{x} = f(x) + g(x) u \]
\[ y = h(x) \]

where \( x \in \mathbb{R}^n, \ y \in \mathbb{R}, \ f \) and \( g \) are smooth vector fields on \( \mathbb{R}^n \) (i.e., \( f(x) \in T_x \mathbb{R}^n, \ x \in \mathbb{R}^n \), and \( h: \mathbb{R}^n \rightarrow \mathbb{R} \) is a smooth function (smooth is understood to mean as differentiable as needed). For simplicity, we assume that \( x = 0 \) is an equilibrium point of the undriven system, i.e., \( f(0) = 0 \), and that \( g(0) \neq 0 \).

If the control objective is tracking, the input-output linearization proceeds as follows: differentiate the output repeatedly until the input appears for the first time on the right-hand side

\[ \dot{y} = L_y h(x), \]
\[ \ddot{y} = L_{y}^2 h(x), \]
\[ \vdots \]
\[ y^{(q)} = L_{y}^q h(x) + L_{y}^q L_{y}^{q-1} h(x) u. \]

\[ \psi(x, \tau) \]
Fig. 3. Simulation results for $y_d(t) = A \cos \frac{\pi t}{5}, A = 1, 2, 3$, using the first approximation.

Fig. 4. Simulation results for $y_d(t) = A \cos \frac{\pi t}{5}, A = 1, 2, 3$, using the second approximation.

Fig. 5. Simulation results for $y_d(t) = A \cos \frac{\pi t}{5}, A = 1, 2, 3$, using the Jacobian approximation.
Here, $L_f h(x)$ stands for the Lie derivative of $h(x)$ along $f$, $L_f^2 h(x)$ stands for $L_f L_f h(x)$ and so on. In (4.2) above, we must have

$$L_x h(x) = L_x L_x h(x) = \cdots = L_x L_x^{n-1} h(x) = 0$$

for $x \in U$ \hspace{1cm} (4.3)

where $U$ is an open neighborhood of the origin. In the event that $L_x L_x^{n-1} h(x) \neq 0$ for $x \in U$, the system is said to have relative degree $\gamma$ and the control law $u = (-L_f h(x) + \nu)/L_x L_x^{n-1} h(x)$ linearizes the system from $u$ to $y$. However, it may happen that $L_x L_x^{n-1} h(x) = 0$ at $x = 0$ but is not identically zero in a neighborhood of $U$ of $x = 0$, i.e., $L_x L_x^{n-1} h(x)$ is a function which is of order $O(x)$ rather than $O(1)$.

Then, the relative degree of the system is not well defined and this input–output linearizing control law is no longer valid.

Since the relative degree fails to exist, we seek a set of functions of the state, $\phi_i(x)$, $i = 1, \ldots, \gamma$, that approximate the output and its derivatives in a special way. The first function $\phi_i(x)$ should approximate the output function, that is

$$h(x) = \phi_i(x) + \psi_i(x, u)$$

where $\psi_i(x, u) = O(x, u)^2$ $\psi_i$ does not depend on $u$, but for notational consistency we include it. Differentiating $\phi_i(x)$ along the system trajectories we get

$$\dot{\phi}_i(x) = L_f \phi_i(x) + L_x \phi_i(x) u.$$ 

If $L_x \phi_i(x)$ is $O(x)$ or of higher order, neglect it (and an $O(x)^2$ part of $L_x \phi_i(x)$ if we so desire) in our choice of $\phi_i(x)$:

$$L_f \phi_i(x) = \phi_i(x) + \psi_i(x, u)$$

where $\psi_i(x, u) = O(x, u)^2$. We continue this procedure with

$$L_f \phi_i(x) = \phi_i(x) + \psi_i(x, u)$$

until at some step $\gamma$, the control term $L_x \phi_i(x)$ is $O(1)$, that is

$$L_f \phi_i(x) = b(x) + a(x) u + \psi_i(x, u)$$

where $a(x)$ is $O(1)$. This procedure motivates the following definition.

**Definition 4.1:** A nonlinear system (4.1) has a robust relative degree of $\gamma$ about $x = 0$ if there exist smooth functions $\phi_i(x)$, $i = 1, \ldots, \gamma$, such that

$$h(x) = \phi_1(x) + \psi_1(x, u)$$

$$L_f \phi_i(x) = \phi_i+x(x) + \psi_i(x, u)$$

for $i = 1, \ldots, \gamma-1$

$$L_f \phi_i(x) = b(x) + a(x) u + \psi_i(x, u)$$

where the functions $\psi_i(x, u)$, $i = 0, \ldots, \gamma$, are $O(x, u)^2$ and $a(x)$ is $O(1)$.

**Remarks:**
- In (4.9), the dependence of $\psi_i(x, u)$ and $u$ has the form $\psi_i(x, u) = \psi_i(x) + \psi_i(x, u)$

$$1 \leq i \leq \gamma$$

where, for $i = 0, \cdots, \gamma$, $\psi_i(x, u)$ is $O(x)^2$ and $\psi_i(x, u)$ is $O(x)$.
- There is considerable latitude in the definition of the $\phi_i(x)$ since each $\phi_i(x)$ may be chosen in a number of ways as long as it is $O(x)^2$.

We now characterize the robust relative degree. First, denote the Jacobian linearization of the system (4.1) about $x = 0$, $u = 0$ by

$$\dot{z} = Az + bu$$

$$y = cz$$

with $A = Df(0)$, $b = g(0)$, and $c = dh(0)$ where $d := \partial \partial x$ and $D := \partial / \partial x$ are derivative operators for scalar functions and maps (e.g., vector fields, coordinate changes), respectively. Suppose that (4.11) has relative degree $\leq i$, hence, $cA^{i-1} b \neq 0$ for some $\gamma \leq n$.

In this case, the following theorem shows that the robust relative degree of the nonlinear system is also well defined.

**Theorem 4.1:** The robust relative degree of the nonlinear system (4.1) is equal to the relative degree of the Jacobian linearized system (4.11) whenever either is defined.

**Proof:** Since $f(0) = 0$ and $d\psi_i(0) = 0$ ($\psi_i(x)$ is $O(x)^2$), it is easy to show that $d\phi_i(x) = cA^{i-1} b$, $1 \leq i \leq \gamma$. Thus, the control coefficients at the equilibrium point are given by $\psi_i(0) = cA^{i-1} b$, $1 \leq i \leq \gamma - 1$, and $a(0) = cA^{\gamma-1} b$. Since $\psi_i(0) = 0$ and $a(0) \neq 0$, it follows that

$$cb = cA^{\gamma-1} b = 0, \cdots, cA^{\gamma-2} b = 0, \cdots, cA^{\gamma-1} b \neq 0.$$ (4.12)

Thus, $\gamma$, the robust relative degree of (4.1), is equal to the relative degree of the Jacobian linearized system (4.11). From this, it is easy to see that $\gamma$ is independent of the choice of the neglected functions $\psi_i(x, u)$ of order $O(x, u)^2$ and is therefore well defined.

An immediate corollary of this theorem is as follows.

**Corollary 4.2:** The robust relative degree of a nonlinear system (4.1) is invariant under a state dependent change of control coordinates of the form $u(x, v) = \alpha(x) + \beta(x)v$ where $\alpha$ and $\beta$ are smooth functions and $a(0) = 0$ while $\beta(0) \neq 0$.

In order to show that the procedure of neglecting the $\psi_i$ terms will produce an approximation to the true system, we first show that the functions $\phi_i(x)$ can be used as part of a (local) nonlinear change of coordinates.

**Proposition 4.3:** Suppose that the nonlinear system (4.1) has robust relative degree $\gamma$. Then the functions $\phi_i(x)$, $i = 1, \cdots, \gamma$, are independent in a neighborhood of the origin.

**Proof:** It is sufficient to show that $d\phi_i$, $1 \leq i \leq \gamma$, are independent at $x = 0$. To check this, simply multiply $D\phi_i(x)$ on the right by $[A^{\gamma-1} b \; A^{\gamma-2} b \; \cdots b]$ to get a lower triangular matrix of all diagonal elements given by $cA^{\gamma-1} b \neq 0$.

This shows that $D\phi_i(x)$ has a rank of $\gamma$ and the proposition is proved.

With the $\gamma$ independent functions $\phi_i(x)$ in hand, we can, by the Frobenius theorem, complete the nonlinear change of coordinates with a set of functions, $\eta_i(x)$, $i = 1, \cdots, n - \gamma$, such that $L_x \eta_i(x) = 0$, $x \in U$. Defining new coordinates $(\xi, \eta)$ by

$$\begin{bmatrix} \xi_1 & \cdots & \xi_\gamma & \eta_1 & \cdots & \eta_{n-\gamma} \end{bmatrix}^T$$

$$:= [\phi_1(x) \cdots \phi_\gamma(x) \eta_1(x) \cdots \eta_{n-\gamma}(x)]^T =: \Phi(x)$$

(4.13)
we rewrite the true system (4.1) as
\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 + \psi_1(x, u) \\
\vdots \\
\dot{\xi}_{\gamma-1} &= \xi_\gamma + \psi_{\gamma-1}(x, u) \\
\dot{\xi}_\gamma &= b(\xi, \eta) + a(\xi, \eta)u + \psi_\gamma(x, u) \\
\dot{\eta} &= q(\xi, \eta) \\
y &= \xi_1 + \psi_0(x, u)
\end{align*}
\] 
\tag{4.14}

where \(q(\xi, \eta)\) expressed in \((\xi, \eta)\) coordinates.

Note that the form (4.14) is a perturbation of the normal form of
Byrnes and Isidori [9], [5] where the terms \(\psi_i(x, u), i = 0, \cdots, \gamma\)
are \(O(x, u)^2\). Thus the control law
\[
u = \frac{1}{a(\xi, \eta)} \left[ - b(\xi, \eta) + \psi\right]
\tag{4.15}
\]
approximately linearizes the system (4.1) from the input \(v\) to the
output \(y\) up to terms of \(O(x, u)^2\).

If the robust relative degree of the system (4.1) is \(\gamma = n\), then the
system (4.1) is approximately full state linearizable. This situation
was investigated by Krener [1] who showed that the system (4.1)
with no output explicitly defined was linearizable to terms of
\(O(x)^n\) if a controllability condition is satisfied and the distribu-
tion (2.5) is order \(p\) involutive, i.e., has a basis, up to terms of
\(O(x)^n\), which is involutive up to terms of \(O(x)^{n-1}\). Equivalently,
these conditions guarantee (through a version of the Frobenius
theorem with remainder \([1]\)) the existence of an output function
\(h(x)\) with respect to which the system has robust relative degree \(n\)
and such that the remainder functions \(\psi_i(x, u)\) are \(O(x, u)^n\). In our
development we are given a specific output function \(y = h(x)\) and a
tracking objective for this output. While [1] deals with conditions
for the existence of approximations, we construct approximate
systems for this purpose. For the ball and beam system, it may be
verified that the involutivity condition is satisfied with \(p = 3\) and
that the system can be input/output and full state linearized up to
terms of order 3.

The choice of the functions \(\Psi_i(x, i = 0, \cdots, \gamma - 1)\), of \(O(x)^2\),
can be used to improve the approximation. One may insist on
choosing these terms to be \(O(x)^\rho\) for some \(\rho \geq 2\). There is less
latitude in the choice of the functions \(\Psi_i(x)\). They must be ne-
eglected if they are \(O(x)\) or higher or neglected if they are not
\(O(1)\) (this determines \(\gamma\)). We cannot in general guarantee that an
approximation of \(O(x, u)^\rho\) for \(\rho > 2\) can be found, although this
may be possible if dynamic feedback is used [10].

In specific applications, the control law (4.15) may produce better
approximations than the Jacobian approximation. Furthermore, the
resulting approximations may be valid on larger domains than the
Jacobian approximation. (Both of these points were illustrated by
the ball and beam system.) We try to make this notion precise by
studying the properties of the approximately linearized system (4.1),
(4.15) on a parameterized family of operating envelopes defined as
follows.

Definition 4.2: We call \(U_\epsilon \subset \mathbb{R}^n\), \(\epsilon > 0\), a family of operating envelopes provided that
\[
U_\epsilon \subset U, \quad \text{where} \quad \delta < \epsilon \quad \text{and} \quad \sup \{ \delta : B_\delta \subset U \} = \epsilon
\tag{4.16}
\]
where \(B_\delta\) is a ball of radius \(\delta\) centered at the origin.

Remarks:
- It is not necessary that each \(U_\epsilon\) be bounded (or compact)
  although this might be useful in some cases.
- Since the largest ball that fits in \(U_\epsilon\) is \(B_\d\), the set \(U_\epsilon\) must get
  smaller in at least one direction as \(\epsilon\) is decreased.

The functions \(\psi(x, u)\) that are omitted in the approximation are
of \(O(x, u)^2\) in a neighborhood of the origin. However, if we are
interested in extending the approximation to (larger) regions, say
of the form of \(U_\delta\), we will need the following definition.

Definition 4.3: A function \(\psi: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}\) is said to be unifor-
metrically higher on \(U_\epsilon \times B_\delta \subset \mathbb{R}^n \times \mathbb{R}\), \(\epsilon > 0\), if for some
\(\sigma > 0\), there exists a monotone increasing function of \(\epsilon\), \(K(\epsilon)\) such
that
\[
|\psi(x, u)| \leq K(\epsilon)|x| + |u| \quad \text{for} \quad x \in U_\epsilon, \quad |u| \leq \sigma.
\tag{4.17}
\]

Remarks:
- If \(\psi(x, u)\) is uniformly higher on \(U_\epsilon \times B_\delta\) then it is
  \(O(x, u)^2\).
- Not all functions \(\psi(x, u)\) that are \(O(x, u)^2\) will be uniformly
  higher order since this definition does not allow for terms that are
  \(O(u)^2\). This does not, however, restrict the choice of \(\psi\) in
  Definition 4.1 since \(O(u)^2\) terms are never present.

Now, we return to the original problem. If the approximate system is
exponentially minimum phase and the error terms \(\psi_i\) in
(4.14) are uniformly higher order on \(U_\epsilon \times B_\delta\), we use the stable
tracking law for the approximate system given by
\[
u = \frac{1}{a(\xi, \eta)} \left[ - b(\xi, \eta) + \psi_\xi \right]
\tag{4.18}
\]
\(\psi = \alpha_{\gamma-1}(\gamma^{n-1} - \xi_\gamma) + \cdots + \alpha_0(\gamma - \xi_0)\) \((\gamma > 2)\)
with \(\alpha_{\gamma-1}, \cdots, \alpha_0\) a Hurwitz polynomial, and
prove the following result.

Theorem 4.4: Let \(U_\epsilon, \epsilon > 0\), be a family of operating envelopes
and suppose that
- the zero dynamics of the approximate system (i.e., \(\dot{\xi} = q(0, \eta)\))
  are exponentially stable and \(q\) is Lipschitz in \(\xi\) and \(\eta\) on \(\Phi(U_\epsilon)\)
  for each \(\epsilon\) and
- the functions \(\psi_i(x, u)\) are uniformly higher order on \(U_\epsilon \times B_\delta\).

Then, for \(\epsilon\) sufficiently small and for desired trajectories with
sufficiently small values and derivatives \((x_0, \dot{x}_0, \cdots, \dot{x}_{n-1})\),
the states of the closed-loop system (4.1), (4.18) will remain bounded
and the tracking error will be \(O(\epsilon)\).

Proof (sketch—see [11] for details): Defining the trajectory error,
\(e \in \mathbb{R}^n\), to be
\[
e = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}^T = \begin{bmatrix} \xi_1 - \gamma_\eta & \xi_2 - \gamma_\eta & \cdots & \xi_\gamma - \gamma_\eta \end{bmatrix}^T
\tag{4.19}
\]
the closed-loop system (4.1), (4.18) (equivalently, (4.14), (4.18))
may be expressed as
\[
\dot{e} = Ae + \psi(x, u(x, \tilde{x}_\eta))
\tag{4.20}
\]
\(\eta = q(\xi, \eta)\)
where \(A\) is an exponentially stable \(\gamma \times \gamma\) matrix in companion
form and \(\tilde{x}_\eta := (x_\eta, \dot{x}_\eta, \cdots, \dot{x}_{\gamma-1})\). The variables \(e\) and \(\eta\)
and (therefore the state \(x\)) are shown to be bounded by showing that
\[
V(e, \eta) = e^TPe + \mu V_2(\eta)
\tag{4.21}
\]
is a Lyapunov function for (4.20). Here \(P > 0\) is chosen so that
\(A^TP + PA = -J\) and \(V_2(\eta)\) is a Lyapunov function for the
exponentially stable zero dynamics \(\dot{\eta} = q(0, \eta)\) (guaranteed to exist by a
converse Lyapunov theorem [12]) and \(\mu\) is a positive constant.

After we have shown that \(e, \eta, \text{and } x\) are locally bounded, it is
not difficult to show that the tracking error will be \(O(\epsilon)\).
Remarks:

- The actual restriction on the class of trajectories that can be tracked is related to how large the functions $\psi_i$ are when the system is approximately producing the output, i.e., when $x$ is close to $y$, $\dot{y}$, etc.

- In certain cases, e.g., the ball and beam, the functions $\psi_i$ may be chosen so that they depend only on the derivatives of the output. In this case, as the simulations show, the main restriction is on the derivatives of the desired trajectory rather than its value.

V. CONCLUSION

In this note, we have presented an approach for the approximate input-output linearization of nonlinear systems, particularly those for which relative degree is not well defined. By designing a tracking controller based on an approximate system, we can achieve tracking of reasonable trajectories with small error. The approximate system is itself a nonlinear system, but is input-output linearizable by state feedback. For the ball and beam system, this approach resulted in a more effective tracking controller than that derived from the standard Jacobian linearization.

REFERENCES


Plant and Controller Reduction Problems for Closed-Loop Performance

Daniel E. Rivera and Manfred Morari

Abstract—Model reduction problems which consider preserving closed-loop performance ($H_\infty$, $H_\infty$, and $\mu$) in the presence of reduction error are developed. These are formulated as weighted multiplicative error problems (for plant reduction) and weighted additive error problems (for controller reduction), with the weight function incorporating explicitly such control information as the desired sensitivity operator bound, the setpoint/disturbance spectrum, and the plant uncertainties. These problems are efficiently solved using the frequency-weighted balanced realization technique. The benefits of these reduction problems are illustrated with examples taken from the control of a binary distillation column.

I. INTRODUCTION

The problem of model reduction is of significant practical importance in control system design, and has been the subject of continuing study since the early 1960's. The vast number of references cited by Genesio and Milanese [1] attests to the fervent activity conducted on this problem.

In the past decade, model reduction by means of balanced realizations and Hankel-norm approximations [2], [3] has stirred great interest within the systems community. Enns has augmented the benefits of balanced realizations by developing a frequency-weighted extension [4] and by proposing weight functions designed to maintain nominal closed-loop stability in the face of reduction error [5]. Anderson and colleagues [6] [7] have developed a frequency-weighted Hankel-norm technique for scalar systems, and have proposed weighted controller reduction problems similar to Enns'. In this note, our objective is to go one step further and formulate weighted reduction problems which maintain desired levels of closed-loop performance (nominal and robust) despite reduction error. The basis for our analysis is the use of linear fractional transformations relating the reduced model to control objectives of interest such as $H_\infty$, $H_\infty$, and the structured singular value ($\mu$) [8], which is a measure of robust stability and performance for systems subject to structured perturbations [9]. These problems are formulated such that they can be efficiently solved by means of the frequency-weighted balanced realization technique.

II. PERFORMANCE OBJECTIVES

The methodology presented applies to linear time-invariant multivariable plants. For the sake of clarity we will initially assume that the full-order plant is asymptotically stable.

A number of control objectives will be of interest in this study. The first is the $H_\infty$ performance objective, defined as

$$\|G\|^2 \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace} \left[G^*G\right] d\omega$$

where $*$ denotes the complex conjugate transpose. We shall also consider the $H_\infty$ performance objective

$$\|G\|_\infty \triangleq \sup_{\omega} \|G(j\omega)\|$$

Finally, we consider model reduction for plants subject to complex norm-bounded perturbations. A large class of perturbations in the classical feedback structure can in turn be represented in terms of $G$ and $\Delta$, where $\Delta$ is a block-diagonal matrix of perturbations. For such a system, the $\mu$ analysis theorem applies.

Theorem RSS (robust stability, structured): Consider the set of perturbations defined by $\Delta = \{ \text{diag} (\Delta_1, \Delta_2, \ldots, \Delta_m) | \Delta_i \in C^{k_i \times k_i}, \text{diag}(\Delta) < 1 \}$. (3)

The plant $G$ is stable for all perturbations described by (3) if and only if

$$\|G\|_\infty \leq 1$$ (4)