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A model of set-theory in which every set of reals is Lebesgue measurable.

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The main result of this paper is the following: One cannot prove in the Zermelo-Fraenkel set theory ZF (which does not contain the axiom of choice) the existence of a set of reals which is not Lebesgue measurable, not even if one uses the principle of dependent choices which allows countably many consecutive choices. The importance of this paper, which contains extremely interesting theorems, is enhanced by its elegant and general technical development which makes it a standard reference for facts about forcing. The theorems of the paper are based on the assumption that the existence of inaccessible cardinals is consistent with ZF. The author represents his results, for technical reasons, as theorems about extensions of model of ZF, but their most interesting aspect is the consistency aspect.

Theorem 1: ZF is consistent with the conjunction of the following statements; (1) The principle of dependent choices; (2) every set A of reals is Lebesgue measurable; (3) every set A of reals has the property of Baire; (4) every uncountable set A of reals contains a perfect subset; (5) let $A = \{A_x : x \in R\}$ be an indexed family of non-empty sets of reals with the reals as the index set, then there are Borel functions h_1 and h_2 mapping R into R such that (a) $\{x|h_1(x) \notin A_x\}$ has Lebesgue measure zero, (b) $\{x|h_2(x) \notin A_x\}$ is of first category. (5) implies that every subset of R^3 has Newtonian capacity. (3) implies the failure of the Hahn-Banach theorem. Once one assumes the axiom of choice then (2)-(5) can be refuted, as is well known. However, it will follow from Theorem 2 that even with the axiom of choice one cannot produce a definable set of reals which is not Lebesgue measurable.

Theorem 2: ZF is consistent with the conjunction of the axiom of choice, the generalized continuum hypothesis and (2)-(5) of Theorem 1 restricted to sets A which are definable from a countable sequence of ordinals.

Theorem 3: ZF is consistent with the conjunction of the axiom of choice, the statement $2^{\aleph_0} =$

$\aleph_\Lambda$, where Λ is any reasonably defined ordinal > 0 and not confinal with ω (such as $2, 3, \omega + 1, \omega_1$, but not $2^{2^{\aleph_0}}$), and with (2)-(5) of Theorem 1 restricted to sets A which are definable from a countable sequence of ordinals.

The main tool of the author is Cohen's forcing. A general concept of genericity is introduced, the particular instances of which are the various forms of Cohen genericity, as well as a new concept of a random real which is vital for the paper, and which is also very useful for other results in set theory.

Reviewed by *A. Lévy*