Optimzation Flow Control

Basic Algorithm and Convergence

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Outline

- Introduction
- Problem Formulation
- The Dual Problem
- Synchronous Distributed Algorithm
- Experimental Results
- Conclusion

Introduction

- Optimization Approach to Flow Control
- A control mechanism is derived.
- Links adjust their Cost
- Sources adjust their Transmission Rates
- Maximize Source Utility Over Transmission Rates

Motivation

- It may not be possible or critical to achieve optimality.

- The optimization framework offers a mechanism to steer the network towards the desirable operating point.

- It is useful to consider flow control schemes as implementations of optimization algorithms. So, modifications to the flow control schemes may be guided by modifications to the optimization problems.

The Primal Problem

- L $= \{1, 2, ..l\}$ is a set of unidirectional links of capacity $c_l, l \in L$
- S $= \{1, 2, ..s\}$ is the set of sources
- $U_s(x_s)$ is the utility function of s where x_s is its transmission rate
- $L(s) \subseteq L$ is the set of links that s uses
- U_s is increasing and strictly concave in its argument
- Our Objective is to choose rates $x = (x_s, s \in S)$ so as to

$$max \textstyle\sum_{s \in S} U_s(x_s)$$

$$\text{subject to } \textstyle\sum_{s \in S(l)} x_s \leq c_l$$

Solving the Primal Problem

- Since the objective function is strictly concave, a unique maximizer exists.
- The objective function is separable in x_s
- But x_s are coupled by constraints on capacity
- Solving the primal directly requires co-ordination

$$max \sum_{s \in S} U_s(x_s)$$

$$\text{subject to } \sum_{s \in S(l)} x_s \leq c_l$$

- This is not possible practically. So, we look at the Dual.

The Primal-Dual Conversion

- Let $A = (a_{ij})_{mXn}, X = (x_i)_n, C = (c_i)_n, B = (b_i)_m, U = (u_i)_m$
- Primal :

$$minC^T X$$

$$\text{subject to } AX = B$$

$$\text{and } X \succeq 0$$

- Dual :

$$maxB^T U$$

$$\text{subject to } A^T U \leq C$$

The Primal-Dual Conversion : An Example

- Primal :

$$min x_1 + 2x_2 + 0x_3 + 0x_4$$

$$\text{such that } -x_1 + x_2 - x_3 = 1$$

$$\text{and } x_1 + x_2 - x_4 = 3$$

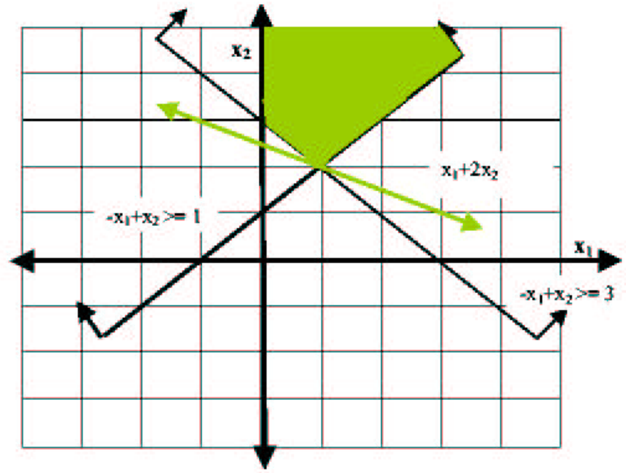
$$\text{and } x_1 \succeq 0, x_2 \succeq 0, x_3 \succeq 0, x_4 \succeq 0$$

- Dual :

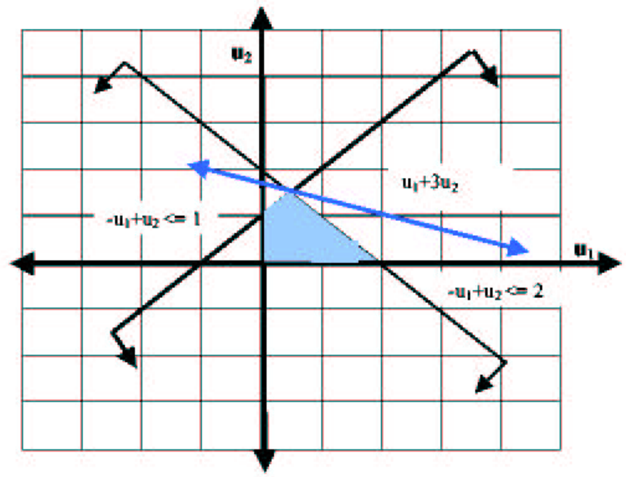
$$max u_1 + 3u_2$$

$$\text{such that } -u_1 + u_2 \leq 1$$

$$\text{and } u_1 + u_2 \leq 1 \text{ and } u_1 + u_2 \leq 2 \text{ and } u_1 \succeq 0, u_2 \succeq 0$$



Primal Problem



Dual Problem

The Dual Problem

- Define the Lagrangian

$$L(x,p) = \sum_s U_s(x_s) - \sum_l p_l (\sum_{s \in S(l)} x_s - c_l)$$

- So, the objective function of the dual problem is

$$D(p) = max L(x,p) = max[\sum_s (U_s(x_s) - x_s \sum_{l \in L(s)} p_l) + \sum_l p_l c_l]$$

- And our dual problem is $min_{p \succeq 0} D(p)$
- The first term is decomposable into S separable sub-problems
- $\sum_{l \in L(s)} p_l$ is all the information a source needs to know
- The dual problem can be solved in a decentralized way !

Solving the Dual Problem

The Dual Problem

$$max[\sum_s (U_s(x_s) - x_s \sum_{l \in L(s)} p_l) + \sum_l p_l c_l]$$

• Since U_s are concave and the constraints are linear, there is no duality gap.

• Therefore the dual optimal prices or the Lagrangian multipliers exist.

• In general, $(x_s(p)$ may not be primal optimal, but if $p^* \succeq$ is dual optimal, then $x_s^*(p)$ is primal optimal, provided it is feasible and the Complementarity Slackness conditions are satisfied.

So, a dual optimal solution gives us a primal optimal solution.

Synchronous Distributed Algorithm

- Solve the Dual Problem using the Gradient Projection Method
- Link Prices are adjusted in the opposite direction to gradient.

$$p_l(t+1) = [p_l t - \gamma \delta D / \delta p_l(p(t))]$$

- Now, $\delta D / \delta p_l(p(t)) = c_l - x^l(p)$ where $x^l(p) = \sum_{s \in S(l)} x_s(p)$
- Therefore, $p_l(t+1) = p_l(t) + \gamma(x^l(t) - c_l)$
- Consistent with the Law of Supply and Demand.
- Again distributed. Can be executed by each link individually.
- The Synchronous algorithm converges to an optimal solution (given a sufficiently small stepsize)

Synchronous Gradient Projection Algorithm

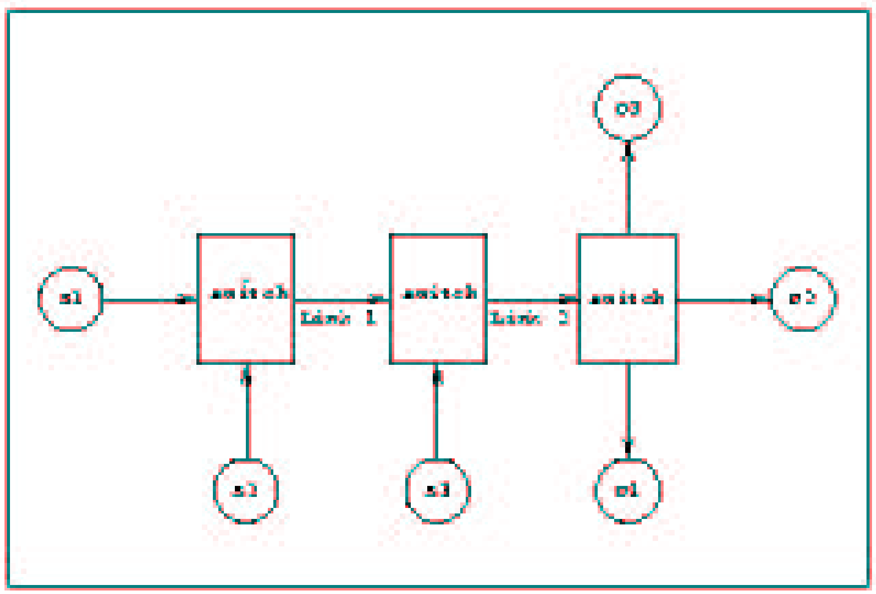
- Link L at times t = 1, 2, ...
  - Receives rates $x_s(t)$ from all sources $s \in S(l)$
  - Computes a new price $p_l(t+1) = p_l(t) + \gamma(x^l(t) - c_l)$ where $x_l(t) = \sum_{s \in S(l)} x_s(t)$
  - Communicates the new price $p_l(t+1)$ to all the sources $s \in S(l)$ that use the link l.
- Source s at times t = 1,2,...
  - Receives from the network $\sum_{l \in L(s)} p_l(t)$
  - Chooses a new transmission rate $x_s(t+1) = x_s(p^s(t))$ where $p^s(t) = \sum_{l \in L(s)} p_l$
  - Communicates the new rate to the links in its path, $l \in L(s)$

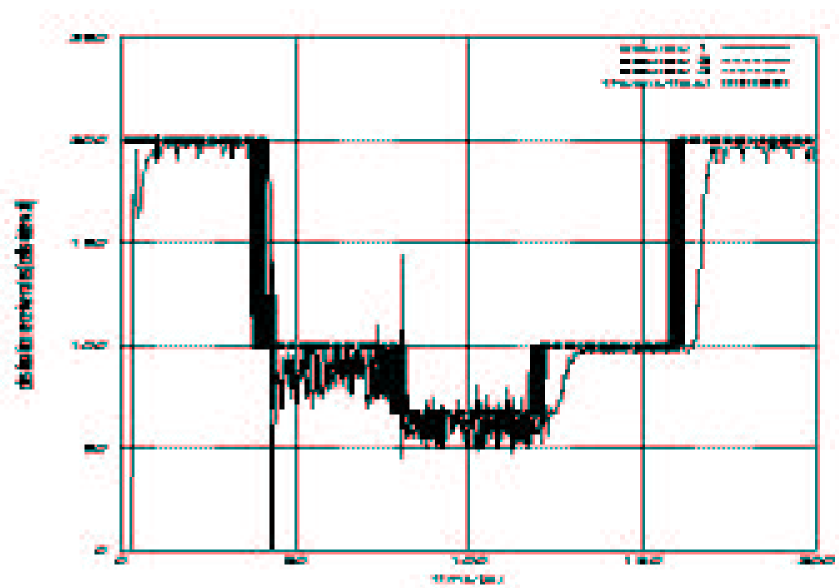
Implementation Issues

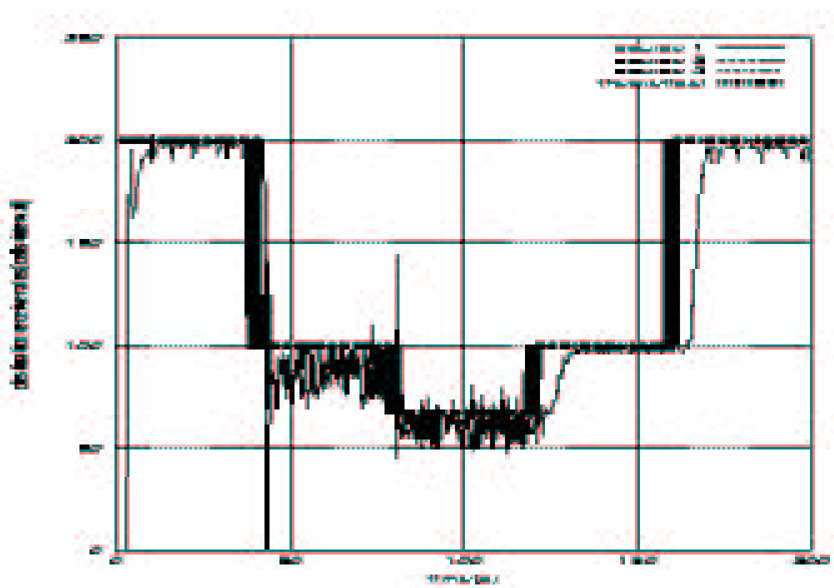
- The communication can be handled using REM.
- Using Newton method which converges faster than the gradient projection method.
- However, synchronous communication is not possible. The authors prove convergence for the Asynchronous Distributed Algorithm as well (if the time between consecutive updates is bounded)
- Pricing may be replaced by signals of congestion.
- This enables provisioning in an ATM like environment.

Experimental Evaluation

- User Level Implementation over UDP
- 3 sources, each tranmitting for 120 secs.
- The starting times are separated by 40secs each
- The Utility function chosen is $a_s log(1 + x_s)$
- The value of a_s is same in the case of homogeneous sources while the value for a_s is double the value of the other flows in heterogeneous case.


switch
Link 1
switch
switch





Summary

- Proposed an Optimization approach to Flow Control
- The Primal problem is centralized.
- Converted the Primal Problem into a Dual Problem.
- The Dual problem is distributed.
- Used Gradient Descent to solve the Dual Problem.
- Showed the feasibility of approach using experiments.
- The scheme is useful for flow control, particularly in ATM like environments.

Comments

- The paper is very well-written.
- It is based on a concrete methodology.
- Requires a lot of mathematical background.
- Experiments are not exhaustive enough.
  - The authors talk of the algorithm converging when the frequency of updates is bounded. They however, do not evaluate it quantitatively.
  - Similarly, they don't take into account the changing nature of utility functions, link bandwidths.

• The basic algorithm cannot be implemented as such on the Internet. It requires the sources to communicate their rates to the links and the links to communicate their prices to the sources. The authors propose a REM (Random Early Marking) scheme to this end.

References

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Thank You !