Some definitions:

**Minimum and Maximum:** When $A$ is a set of real numbers, one designates by $\min A$ the smallest number in the set. For instance, $\min\{1,2,3.1,0.7,2.1\} = 0.7$. The number $\max A$ is defined similarly and $\max A = 3.1$.

**Infimum and Supremum:** Some sets do not have a smallest element. For instance, $(0, 1]$ has no smallest element, so that $\min (0, 1]$ is not defined. To take care of such cases, one defines $\inf A$ to be the largest number that is smaller than or equal to all the elements of $A$. For instance, $\inf (0, 1] = 0$ because $0$ is smaller than or equal to any $x \in (0, 1]$. Note that $\inf [0, 1] = \min [0, 1] = 1$. This number $\inf A$ is called the greatest lower bound of $A$ and is sometimes denoted by $g.l.b. A$, although we will not use this notation. The number $\sup A$ is defined similarly and $\sup (0, 1) = 1$.

**Problem I.1:** Let $A = [0, 11] \times [0, 10]$ and $B = \{(x, y) \in [0, \infty)^2 | x^2 + y^2 \geq 144\}$.

a. Find $\min \{x \mid (x, y) \in A \cap B\}$.

b. Find $\min \{y \mid (x, y) \in A \cap B\}$.

c. Find $\sup \{x + 2y \mid (x, y) \in A \setminus B\}$.

d. Find $\min \{x^4 + 2y^4 \mid (x, y) \in A \cap B\}$.

a. $\sqrt{44}$

b. $\sqrt{23}$

c. The objective function $x + 2y$ increases in $x$ and $y$. So it cannot be maximized at an interior point of the set because moving towards up or right always increases the objective function. Furthermore the maximum must be along the circular portion of the set since moving right on the linear portion of the top boundary increases the objective and moving up on the linear portion of the right boundary increases the objective. You could use Lagrange multipliers or you could recall a theorem from real analysis that says that a continuous function over a closed interval attains its extrema in the interval. The maximum value is attained at the point $(\sqrt{44}, 10)$. The objective at this point is $\sqrt{44} + 20$.

d. Again the objective function increases in $x$ and $y$. So the minimum must be on the lower circular boundary. Again once we restrict our attention to the boundary, we have a constrained optimization problem. Either use Lagrange multipliers or substitute the constraint $x^2 + y^2 = 144$ into the objective $x^4 + 2y^4$ and find the minimum of this function over the closed interval $[\sqrt{23}, 10]$. Remember the minimum may occur either at the end points or at a point where the derivative is zero within $[\sqrt{23}, 10]$. The objective is minimized at $(11, \sqrt{23})$ and at this point takes on the value of 1179.

**Problem I.2:** For each set

a. Find $\inf \{x \mid x$ is rational and $x > \sqrt{2}\}$. 
b. Find
\[ \inf \left\{ \frac{1}{x} \mid x > 2 \right\}. \]

a. The inf of this set is \( \sqrt{2} \). To prove this we check to see if \( \sqrt{2} \) satisfies the definition. The definition has two requirements. First, \( \sqrt{2} \) must be a lower bound of the set. This is clearly true. Second, for any \( x \in \mathbb{R} \) where \( x > \sqrt{2} \), \( x \) must not be a lower bound of the set - i.e. there exists at least one element in the set less than \( x \). This is the rigorous meaning of an inf being the greatest lower bound. If you take anything bigger you no longer have a lower bound.

So take any \( x \in \mathbb{R} \) that is strictly greater than \( \sqrt{2} \). Between any two real numbers there exists at least one rational number. This is what is meant when we say that the rationals are dense in the reals. This is one of the first theorems you would prove in a course like math 104. You may assume that it is true for this class.

We now show this formally. Consider an integer \( \frac{1}{q} \) and \( \frac{1}{n} \) will be strictly less than \( y \) and will also be in the set in question. So \( 0 \) is the inf of the set in question by definition. \( \square \)

b. The inf of this set is 0. Again the proof consists in nothing more than checking the definition. 0 is clearly a lower bound of this set. Consider any \( y > 0 \). Can \( y \) be a lower bound of the set? No, it cannot because I can take any integer greater than \( \max \{2, \frac{1}{y}\} \) and \( \frac{1}{n} \) will be strictly less than \( y \) and will also be in the set in question. So \( 0 \) is the inf of the set in question by definition. \( \square \)

**Problem I.3:** Let \( A \) be a nonempty subset of \( (-\infty, 5] \).

a. Show that \( \alpha = \sup A \) is a well-defined number that is less than or equal to 5.

b. Explain that one can select a sequence \( \{x_1, x_2, x_3, \ldots\} \) of elements of \( A \) that approach \( \alpha \). Precisely, for any arbitrarily chosen \( \epsilon > 0 \), there is some integer \( n(\epsilon) \) such that \( |x_n - \alpha| \leq \epsilon \) for all \( n \geq n(\epsilon) \). We say that \( x_n \) converges to \( \alpha \) and we write

\[ \lim_{n \to \infty} x_n = \alpha. \]

a. The reals have a property called the “Least upper bound property.” As you might guess this means that for any non-empty set \( E \subset \mathbb{R} \) that is bounded from above, \( \sup E \) exists in the reals. When we say that \( \alpha = \sup A \) is a well-defined number we mean that this real number \( \sup A \) exists in the reals. This seems silly at first glance. Why wouldn’t it exist in the reals? But it is not something you can take for granted. In the previous problem part a, there is no \( \inf \) or greatest lower bound of the set in the **rationals** since the greatest lower bound - \( \sqrt{2} \) - is an irrational. Thus you might say that the rationals do not have the “Greatest lower bound property.” BTW the rationals also do not have the “Least upper bound property” either. So since \( A \) is non-empty and bounded above by 5, \( \alpha = \sup A \) must exist and you can find it in the set of reals.

So we still need to show that it is less than or equal to 5. This can be shown by a small proof by contradiction. Suppose \( \alpha > 5 \). Then 5 cannot be an upper bound of \( A \) because then \( \alpha \) would not be the **least** upper bound of \( A \). Recall the definition of sup. First \( \alpha \) must be an upper bound of \( A \), and second if a number \( x \) is less than \( \alpha \) then \( x \) cannot be an upper bound of \( A \) - i.e. you can find an \( y \in A \) such that \( x < y \leq \alpha \). But we know 5 is an upper bound of \( A \). Contradiction. Thus we must have \( \alpha \leq 5 \). b. All we need to do for this part is give an example of how this can be done. Actually problem 4 hints at one example. Consider the monotonically increasing sequence \( A = \{x_n, n \geq 1 \mid x_n = \alpha - \frac{1}{n}\} \).

It is pretty clear by inspection that this sequence should converge to \( \alpha \). We now show this formally. Given any arbitrary \( \epsilon > 0 \) we need to be able to find a positive integer \( N \) such that for all \( n > N \), \( |x_n - \alpha| \leq \epsilon \). Consider an integer \( N > \frac{1}{\epsilon} \). For all \( n > N \) we have
\[ \frac{1}{n} < \epsilon \]
\[ \frac{1}{n} \mid < \epsilon \]
\[ |x_n - \alpha| < \epsilon \]  

Thus by definition \( \lim_{n \to \infty} x_n = \alpha \). \( \Box \)

**Problem I.4:** Let \( \{x_n, n \geq 1\} \) be a collection of real numbers such that
\[ x_n \leq x_{n+1} \leq 12 \text{ for all } n \geq 1. \]

Show that there is some number \( \alpha \leq 12 \) such that
\[ \lim_{n \to \infty} x_n = \alpha. \]

The point of this problem is that a monotonically non-decreasing sequence that is bounded from above converges to its supremum. The supremum requires a set with which to be associated. So let \( S = \{x_n, n \geq 1\} \) denote the set of the sequence values. \( S \) is bounded above and non-empty. By the “Least upper bound property” of the reals \( \sup S \) exists in \( \mathbb{R} \).

So I claim the \( \alpha \) such that \( \lim_{n \to \infty} x_n = \alpha \) is \( \sup S \). To show convergence of a sequence I need to show for any \( \epsilon > 0 \), I can find an integer \( N \) such that for all \( n > N \) I have \( |x_n - \alpha| \leq \epsilon \). In words this means that after the \( N^{th} \) real number in the sequence, all subsequent numbers in the sequence are within \( \epsilon \) of \( \alpha \). Another way of saying this is that the sequence is not within \( \epsilon \) of \( \alpha \) infinitely many times.

I chose to show this by proof by contradiction. So suppose that \( x_n \) does not converge to \( \alpha \). Then there exists at least one \( \epsilon > 0 \) such that there is no \( N \) for which for all \( n > N \), \( |x_n - \alpha| \leq \epsilon \). In words this says that the sequence is not within \( \epsilon \) of \( \alpha \) infinitely many times. This does not preclude the sequence from being within \( \epsilon \) of \( \alpha \) infinitely many times (When can this happen?). Notice also the negation of the convergence statement. Whenever you have a proposition that is true for all \( \epsilon > 0 \), the negation of that statement is that the proposition is not true for at least one \( \epsilon > 0 \).

Okay so denote this rogue \( \epsilon \) as \( \epsilon_r \). So for some integer \( m \), \( |x_m - \alpha| > \epsilon_r \). I am guaranteed to find at least one such \( m \) because I assumed the sequence does not converge and hence there are infinitely many sequence values not within \( \epsilon_r \) of \( \alpha \) to choose from. Since \( \alpha \) is an upper bound of \( S \) - recall \( \alpha = \sup S \) and the definition of sup - we actually have for this \( m \) that \( \alpha - x_m > \epsilon_r \). Rearranging this we obtain \( x_m < \alpha - \epsilon_r \). But \( \alpha \) is the least upper bound of \( S \). Consequently \( \alpha - \epsilon_r \) is not an upper bound of \( S \). That means there exists an integer \( N \) such that \( x_N > \alpha - \epsilon_r \) which implies that \( \alpha - x_N < \epsilon_r \). What can we say for \( x_n \) such that \( n \geq N \)? The \( x_n \)'s are a non-decreasing sequence. So:

\[ x_N \leq x_n \]
\[ -x_N \geq -x_n \]
\[ \epsilon_r > \alpha - x_N \geq \alpha - x_n \geq 0 \]
\[ |\alpha - x_n| < \epsilon_r \]  

So for all \( n \geq N \), \( x_n \) is within \( \epsilon_r \) of \( \alpha \). But this means that at most \( N - 1 \) sequence elements are not within \( \epsilon_r \) of \( \alpha \). Contradiction! There is no such \( \epsilon_r \), and the sequence converges to the supremum of \( S \). \( \Box \)
Problem I.5: A 32-card deck the cards 7, 8, 9, 10, Ace, Jack, Queen, King of four suits (clubs, spades, hearts, and diamonds). How many different subset sets of five cards are there with two pairs (and no triple)? A pair is a set of two cards with the same value (e.g., two queens or two 8s).

Let’s count how many different ways to assemble this hand. Here are two equivalent ways. First pick the face-value - i.e. 10, Ace, King, etc - that the two pairs can take on. There are eight possible face-values, and I need to choose two distinct face values since picking the same face value for two pairs yields a four of a kind which is an instance of a three of triple. I can do this \( \binom{8}{2} \) ways. I then need to count the number of ways to construct the first pair from a choice of four suite values. I can do this \( \binom{4}{2} \) ways. The number of ways to construct the second pair from a choice of four suite values is \( \binom{4}{2} \) as well. I now need to pick one more card to finish my hand. I can’t pick any of the remaining cards because I might end up with a triple. Thus I exclude all cards with the same face-value as those in the two pairs chosen. I choose one card out of 32 – 8 = 24 remaining cards - \( \binom{24}{1} \).

Number of ways to pick said hand = \( \binom{8}{1} \binom{4}{2} \binom{4}{2} \binom{24}{1} \) \hspace{1cm} (3)

Another way to construct the hand is as follows. We clearly need three distinct face-values - two assigned to a pair and one to the singleton. We have \( \binom{8}{3} \) ways of picking three face-values. We then need to assign two of those face values to two pairs. There are three face-values and two pairs so we can do this \( \binom{3}{2} \) ways. We then need to count the number of ways to generate the first pair of the given face-value from a choice of four suites - \( \binom{4}{1} \). We have the same number of ways to construct the second pair. Finally there are \( \binom{4}{1} \) ways to choose the singleton from the four suites.

Number of ways to pick said hand = \( \binom{8}{3} \binom{3}{2} \binom{4}{2} \binom{2}{2} \binom{1}{1} \) \hspace{1cm} (4)

Problem I.6: Calculate

\[
\int_0^1 x^n e^{-x} dx
\]

where \( n \) is any integer.

For any negative \( n \) the integral diverges. It is not hard to show that for \( n = 0 \) that the integral is \( 1 - e^{-1} \). To determine the value of the integral when \( n \) is positive, use integration by parts. First let us index the original integral by \( n \): \( \Gamma(n) \). Then we get for \( n \geq 1 \):

\[
\Gamma(n) = \int_0^1 x^n e^{-x} dx = -e^{-1} + n \int_0^1 x^{n-1} e^{-x} = -e^{-1} + n \Gamma(n - 1)
\]

If you re-apply integration by parts a few more times it is not hard to show the following:
\[
\Gamma(n) = -e^{-1}(1 + n + n(n - 1) + \ldots + n!) + n!\Gamma(0)
\]
\[
= -e^{-1}n!\left(\frac{1}{n!} + \frac{1}{(n-1)!} + \ldots + \frac{1}{2!} + \frac{1}{1!}\right) + n!(1 - e^{-1})
\]
\[
= -e^{-1}n!\left(\sum_{i=1}^{n} \frac{1}{i!}\right) + n!(1 - e^{-1})
\]
\[
= n!(1 - e^{-1}\left(\sum_{i=0}^{n} \frac{1}{i!}\right))
\]

Putting all three cases together:
\[
\int_{0}^{1} x^n e^{-x} dx = \begin{cases} 
\infty & \text{if } n < 0, \\
n!(1 - e^{-1}\left(\sum_{i=0}^{n} \frac{1}{i!}\right)) & \text{if } n \geq 0.
\end{cases}
\]

Problem I.7: Prove by induction that
\[
1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n + 3n^2 + 2n^3}{6}.
\]

Proof by induction is a special type of constructive proof. Obviously one way to show the claim is true is by actually proving the claim for each and every value of \( n = 1, 2, \ldots \) This would take too long. Instead we prove the claim for one case and then prove subsequent cases are true as long as there is at least one case is true. So again the two steps are:

1. Verifying the claim for a base case upon which other cases will be proved.
2. Proving other cases will be true as long as at least one case is true.

Showing the second step means that the claim is true for other cases as long as there is at least one case that is true. Showing the first step means that the claim is true for at least one case. Together they show that the claim is true for all cases.

1. Here the base case is \( n = 1 \). (Can I start with a different base case? What are the merits of starting at \( n = 1 \)?) So is the claim true for \( n = 1 \)? The LHS for \( n = 1 \) is \( 1^2 \). The RHS is \( \frac{1+3 \times 1^2 + 2 \times 1^3}{6} \) which is just 1. Yes, the base case is true.
2. Now for the inductive step I will show that if the claim is true for \( n = k \) then it is true for \( n = k + 1 \). So select an arbitrary \( k \) such that (saakst)
\[
\sum_{i=1}^{k} i^2 = \frac{k + 3k^2 + 2k^3}{6}
\]

This is the induction hypothesis. Notice this is not going to do me any good if there is no arbitrary \( k \) such that the claim is true. But I am guaranteed by the first step that there is at least one such \( k \) such that the claim is true - namely \( k = 1 \).

Consider now \( \sum_{i=1}^{k+1} i^2 \):

\[
\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k + 1)^2
\]

\[
= \frac{k + 3k^2 + 2k^3}{6} + (k + 1)^2
\]

\[
= \frac{6 + 13k + 9k^2 + 2k^3}{6}
\]

\[
= \frac{(1 + 3 + 2) + (1 + 6 + 6)k + (3 + 6)k^2 + 2k^3}{6}
\]

\[
= \frac{(k + 1) + 3(1 + 2k + k^2) + 2(1 + 3k + 3k^2 + k^3)}{6}
\]

\[
= \frac{(k + 1) + 3(k + 1)^2 + 2(k + 1)^3}{6}
\]

The equality denoted by (a) is true by the induction hypothesis. \qed