**Problem VI.1:** Let \((X, Y)\) be the coordinates of a point picked uniformly in a circle centered at the origin and with radius 1. Calculate

a. \(E[X \mid Y \geq a]\) for \(a \in (-1, +1)\);

b. \(E(|X - Y|^2)\);

c. \(E[X \mid X \geq Y]\);

d. \(E[X \mid X + Y]\).

**Problem VI.2:** Assume that \(X\) and \(Y\) are independent and uniformly distributed in \([0, 1]\). Calculate \(E[X|X^2 + Y^2]\).

**Problem VI.3:** Let \((X, Y)\) designate the coordinates of a point picked uniformly on the circumference of a circle with radius one centered at the origin. Calculate \(E[X + Y|X + Y > 0]\).

**Problem VI.4:** Let \(X, Y, Z\) be independent and exponentially distributed with mean 1. Calculate \(E[X|X + Y + Z]\).

**Problem VI.5:** Assume that \(X\) and \(Y\) are independent and uniformly distributed in \([0, 1]\). Calculate \(E[(X + Y)^3|X]\).

**Problem VI.6:** Assume that an event \(A\) happens with an unknown probability \(p\) in a single trial. The value of \(p\) is picked uniformly in \([0, 1]\). Find the expected value of \(p\) given that the event \(A\) occurs \(a\) times out of \(a + b\) independent replications of the trial.

Apply this result to the following example. You flip a coin that has an unknown probability of yielding H. The first \(n\) flips yield H. What is the probability that the next flip will also yield H?

*Note:* When \(a\) and \(b\) are integers,\
\[
\int_0^1 x^{a-1} (1 - x)^{b-1} \, dx = \frac{(a-1)!(b-1)!}{(a+b-1)!}.
\]

**Problem VI.7:** Let \(X\) and \(Y\) be independent and uniformly distributed in \([0, 1]\). Calculate \(E[X|XY]\).

**Problem VI.8:** Let \(X\) be a random variable with mean \(\mu\) and finite variance \(\sigma^2\). Show that the value of \(a\) that minimizes \(E[(X - a)^2]\) is \(a = \mu\).

**Problem VI.9:** Given \(Y\), let \(X_1, X_2, \ldots\) be independent and uniformly distributed on \([0, Y]\).

a. Assume that \(Y\) is uniformly distributed on \([0, A]\). Find \(E[Y \mid \max\{X_1, \ldots, X_n\}]\).

b. Repeat the problem, assuming that \(Y\) is exponentially distributed with mean \(A\).