Problem 1. a. The sample space is the space of all possible poker hands (5 card hands).
One way to write this is as follows: Denote a card $c_i$ as a pair $c_i = \{(x_i, y_i)| x_i \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}, y_i \in \{\spadesuit, \heartsuit, \clubsuit, \diamondsuit}\}$. Then the sample space can be denoted as $\{(c_1, c_2, c_3, c_4, c_5)| c_i \neq c_j \forall i \neq j\}$.

For calculations in part b, we will want to know the size of this sample space. Since we assume that all poker hands are equally likely, finding the probabilities in part b is just counting the number of ways for the events to occur divided by the size of the sample space. Here the size of the sample space is $\binom{52}{5}$ since we are interested in combinations of distinct cards.

b. i. The event of a flush can be denoted as $\{(c_1, c_2, c_3, c_4, c_5)| c_i \neq c_j \forall i \neq j; and \ y_i = y_j \forall i, j\}$. Notice that the suit $y_i$ is the same for all $c_i$ since a flush occurs when all the cards in the hand are of the same suit.

There are 4 different suits from which we need to choose 1, and there are 13 possible cards in that chosen suit from which we want a combination of 5. $P(\text{flush}) = \binom{13}{1}\binom{4}{5}\binom{52}{5}$

ii. The event of exactly one pair can be denoted as $\{(c_1, c_2, c_3, c_4, c_5)| c_i \neq c_j \forall i \neq j; \exists i \neq j \text{ such that } x_i = x_j \text{ and } \forall m, n \neq i \ x_n \neq x_m\}$

There are 13 different numbers to choose from for the possible pairs. There are 4 possible cards with the same number from which we can choose the pair. Then we must choose the remaining 3 cards must be chosen such that they all have different numbers from each other and from the pair. $P(\text{exactly one pair}) = \binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{52}{5}$

iii. The event of exactly two pair can be denoted as $\{(c_1, c_2, c_3, c_4, c_5)| c_i \neq c_j \forall i \neq j; \exists i \neq j \neq k \neq l \neq m \text{ such that } x_i = x_j \neq x_k = x_l \text{ and } x_m \neq x_i, x_l\}$

There are 13 different numbers from which we choose 2 to be the pairs. For each pair, there are 4 possible cards from which we can choose. Then we must choose the remaining card. $P(\text{exactly two pair}) = \binom{13}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}\binom{52}{5}$

iv. The event of exactly three of a kind can be denoted as $\{(c_1, c_2, c_3, c_4, c_5)| c_i \neq c_j \forall i \neq j; \exists i \neq j \neq k \text{ such that } x_i = x_j = x_k \text{ and } \forall m, n \neq i, j \ x_n \neq x_m\}$
There are 13 different numbers to choose the 3 of a kind. There are 4 possible cards from which we need to choose 3. The remaining 2 cards need to be of different numbers than each other and the three of a kind. \( P(\text{exactly one three of a kind}) = \frac{\binom{13}{3}\binom{4}{2}(\frac{4}{5})^2}{\binom{52}{5}} \).

v. The event of a full house (three of a kind and one pair) can be denoted as \( \{(c_1, c_2, c_3, c_4, c_5)|c_i \neq c_j \forall i \neq j; \exists i \neq j \neq k \neq l \neq m \text{ such that } x_i = x_j = x_k \neq x_l \} \).

There are 13 different numbers from which we can choose the three of a kind. For this number, there are 4 possible choices from which we need to choose 3. There are 12 numbers remaining from which we can choose have the pair. There are 4 possible choices from which we need to choose 2. \( P(\text{full house}) = \frac{\binom{13}{3}\binom{4}{2}\binom{4}{2}}{\binom{52}{5}} \)

**Problem 2.**

a. Let \( E_i \) denote the event that the \( i \)th student receives the correct assignment, and let \( E \) denote the event that none of the students receive the correct assignment.

\[
P(E) = 1 - P(\text{at least one student receives the correct assignment})
\]

\[
= 1 - P(\bigcup_{i=1}^{N} E_i)
\]

\[
= 1 - \sum_{i=1}^{N} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} \cap E_{i_2}) + \ldots
\]

\[
+ (-1)^{n+1} \sum_{i_1 < i_2 < \ldots < i_n} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_n}) + \ldots
\]

\[
+ (-1)^{N+1} P(E_1 \cap E_2 \cap \ldots \cap E_n)
\]

Notice we used Corollary 6 on page 34 of the textbook.

Consider the \( n \)th term \( \sum_{i_1 < i_2 < \ldots < i_n} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_n}) \). These are the events where \( n \) of the students receive the correct assignment. There are \( \binom{N}{n} \) terms in that sum.

The probabilities \( P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_n}) = \frac{(N-n)!}{N!} \) since there are \( N! \) ways to hand back the assignments and there are \( (N-n)! \) ways for a fixed \( n \) students to receive their correct assignments. Hence \( \sum_{i_1 < i_2 < \ldots < i_n} P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_n}) = \binom{N}{n} \frac{(N-n)!}{N!} = \frac{1}{n!} \).

Thus \( P(E) = \sum_{n=2}^{N} \frac{(-1)^n}{n!} \).

b. For \( N \) large, the probability is approximately equal to \( 1/e \) which is approximately 0.36788.

c. First fix a particular set of \( k \) students. The number of ways in which these and only these \( k \) students can receive their assignments is equal to the number of ways in which the other \( N - k \) students can receive the remaining assignments in such a way that none of them receives his own. From part a, \( \sum_{n=0}^{N-k} \frac{(-1)^n}{n!} \) is the probability that none of the \( N - k \) students receive their own assignment. So the number of ways for none of the \( N - k \) students to receive their own assignment is \( (N-k)! \sum_{n=0}^{N-k} \frac{(-1)^n}{n!} \).

There are \( \binom{N}{k} \) possible selections of groups of \( k \) students. Hence there are \( \binom{N}{k}(N- \)
Figure 1: The shaded regions are the areas where $(c_1, c_2)$ is such that the largest piece is twice as large as the other two.

$$k)! \sum_{n=0}^{N-k} \frac{(-1)^n}{n!} \text{ ways in which } k \text{ students receive their correct assignments. The desired probability is then } \frac{(N-k)! \sum_{n=0}^{N-k} \frac{(-1)^n}{n!}}{N! k! \sum_{n=0}^{N-k} \frac{(-1)^n}{n!}} = \frac{1}{k!} \sum_{n=0}^{N-k} \frac{(-1)^n}{n!}.
$$

**Problem 3.** Require 3 cuts to get 3 pieces. Notice that the size of the pieces is proportional to the arc length of the piece.

The position of the first cut doesn’t matter. After the first cut, we can think of the problem as 'unrolling’ the circumference of the cake into a line. Let the circumference be of the length L. The problem is then simplified to cutting the length L into 3 pieces and then finding the probability that the longest piece is twice as large as the sum of the other 2.

Let $(c_1, c_2) \in [0, L] \times [0, L]$ denote the positions of the first and second cuts respectively.

First consider the case when $c_1 < c_2$. The size of the three pieces is proportional to $x = c_1$, $y = c_2 - c_1$, and $z = L - c_2$. Then the event that the largest piece is twice as large as the other two combined (call this $E$) is the following $E = X \cup Y \cup Z$, where

- $X = \{(c_1, c_2) | x \geq 2(y + z)\} = \{(c_1, c_2) | c_1 \geq \frac{2L}{3}\}$,
- $Y = \{(c_1, c_2) | y \geq 2(x + z)\} = \{(c_1, c_2) | c_2 - c_1 \geq \frac{2L}{3}\}$,
- $Z = \{(c_1, c_2) | z \geq 2(x + y)\} = \{(c_1, c_2) | c_2 \leq \frac{L}{3}\}$.

Similarly find $X, Y, Z$ for the case when $c_2 \leq c_1$. Plot the results as in Figure 1.

The probability is proportional to the area of these events. Hence $P(\text{largest piece is twice as large as the other two}) = \frac{1}{3}.$
Problem 4. a. Let $E_n$ denote the event that a head first appears in the $n$th toss. Notice that these events $E_n$ are disjoint. Have that $P(E_n) = \frac{1}{2^n}$ since there is only one way for a head to appear in the $n$th toss (that is when we have tails in the first $n-1$ tosses and then a head in the $n$th toss) and there are a total of $2^n$ possibilities for the outcomes of the first $n$ tosses.

Let $E$ denote the event that a head is bound to turn up sooner or later, that means that one of the events $E_n$ occurred for some $n$. Hence $E = \bigcup_{n=1}^{\infty} E_n$.

$$P(E) = P(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

b. Let $s = \{s_1, s_2, \ldots, s_n\}$ be any finite sequence. We want to show that this sequence will show up in an infinite sequence of coin tosses. Consider the infinite sequence of coin tosses arranged in disjoint groups of consecutive outcomes, each group being of length $k$. There is a probability of $\frac{1}{2^k}$ that any given one of these is $s$, independently of the others.

The event $\{\text{one of the first } n \text{ such groups is } s\}$ is a subset of the event $\{s \text{ occurs in the first } nk \text{ tosses}\}$. Hence we have that

$$P(s \text{ turns up eventually}) = \lim_{n \to \infty} P(s \text{ occurs in the first } nk \text{ tosses})$$

$$\geq \lim_{n \to \infty} P(s \text{ occurs as one of the first } n \text{ groups})$$

$$= 1 - \lim_{n \to \infty} P(\text{none of the first } n \text{ groups is } s)$$

$$= 1 - \lim_{n \to \infty} 1 - \left(\frac{1}{2^k}\right)^n$$

$$= 1$$

Problem 5. Let $A$ and $B$ be events with probabilities $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$.

a. $P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1 = \frac{1}{12}$ since $P(A \cup B) \leq 1$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $P(A \cap B) \leq \min\{P(A), P(B)\} = \frac{1}{3}$.

b. Pick a number at random from $\{1, 2, \ldots, 12\}$. Take $A = \{1, 2, \ldots, 9\}$ and $B = \{9, 10, 11, 12\}$. Then $A \cap B = \{9\}$ so $P(A) = \frac{3}{4}$, $P(B) = \frac{1}{3}$, and $P(A \cap B) = \frac{1}{12}$.

c. Pick a number at random from $\{1, 2, \ldots, 12\}$. Take $A = \{1, 2, \ldots, 9\}$ and $B = \{1, 2, 3, 4\}$. Then $A \cap B = \{1, 2, 3, 4\}$ so $P(A) = \frac{3}{4}$, $P(B) = \frac{1}{3}$, and $P(A \cap B) = \frac{1}{3}$.

Problem 6. We need to find 1-1 correspondences for the elements of sets $A$ and $B$ to a countable set and an uncountable set respectively.

One possible 1-1 correspondence would be as follows. Let $s$ be an element of $A$ or $B$ (of course, $s$ would be finite if it was from $A$). Define a 1-1 correspondence by having
the $i$th term in $0.s_1, s_2, s_3, \ldots$ be 1 if $s_i \in s$ and be 0 otherwise. For example, $1,3,8$ would be mapped to $0.10100001$. Notice that elements of $A$ are finite so the digit they are mapped to is finite.

Thus elements $A$ are mapped to are some subset of rational numbers which is countable. Thus $A$ is countable.

Elements of $B$ can be infinite. So the elements $B$ are mapped to may be infinite. If we let $0.s_1, s_2, s_3, \ldots$ denote the binary expansion of numbers between $[0,1]$ we see that the above map defines a 1-1 correspondence between $B$ and $[0,1]$, which is uncountable. Hence $B$ is uncountable.