Problem 1. A random variable $X$ has pdf

$$f_X(x) = \begin{cases} \frac{c}{2}(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

a. Find $c$.
b. Find $P(\frac{1}{2} \leq X \leq \frac{3}{4})$.
c. Find $F_X(x)$.

Problem 2. Express the distribution functions of the following in terms of the distribution function $F$ of the random variable $X$.

a. $X^+ = \max\{0, X\}$
b. $X^- = -\min\{0, X\}$
c. $|X|$
d. $-X$

Problem 3. A dart is flung at a circular dartboard of radius 3. Suppose that the probability that the dart lands in some region $A$ is just proportional to the area of that region, $P(A) = \frac{\text{area}(A)}{9\pi}$.

For each of the following scoring systems:

i. Determine and plot the distribution function of the score.
ii. Calculate the expected value of the score.

a. Consider the following scoring system. The dartboard is partitioned into three concentric circles centered at the origin with radii 1, 2, and 3. These circles divide the target into three annuli $A_1, A_2,$ and $A_3$ where $A_k = \{(x, y)|k-1 \leq \sqrt{x^2 + y^2} < k\}$. The resulting score $X$ is the random variable given by $X(\omega) = k$ whenever $\omega \in A_k$.

b. Let the score be an amount equal to the distance between the hitting point $\omega$ and the center of the dartboard. The score $Y$ is a random variable given by $Y(\omega) = \sqrt{x^2 + y^2}$ if $\omega = (x, y)$.
c. Now suppose that the player fails to hit the dartboard altogether with probability $p$. If he is successful, then we suppose that the distribution of the hitting point is proportional to the area of the region. If he hits the dartboard, then he receives a score equal to the distance between the hitting point and the center; if he misses the dartboard, then he receives a score of 4. Let $Z$ denote his score.

**Problem 4.** Suppose we put $m$ balls randomly into $n$ boxes. A box can hold more than one ball. What is the expected number of empty boxes? (Hint: What is the probability that the first box is empty?)

**Problem 5.** A cereal company is running a promotion for which it is giving out a toy with every box of cereal. There are $n$ different types of toys. However, when you purchase a box of cereal, you do not know beforehand which toy is contained in the box. Assume that each box is equally likely to contain any one of the $n$ toys independently from box to box. Each box of cereal costs $C$. Find the expected amount of money you have to spend to collect all $n$ toys.

**Problem 6.** Consider the unit square $[0, 1] \times [0, 1]$. Pick a point $P$ uniformly on that square. Let $\theta$ denote the angle made between the x-axis and the vector formed by the origin and point $P$. Find the cdf, pdf, and expectation of $\theta$.

**Problem 7.** The noise voltage in an electric circuit can be modeled as a Gaussian random variable with mean equal to zero and variance equal to $10^{-8}$.

a. What is the probability that the value of the noise exceeds $10^{-4}$? What is the probability that it exceeds $4 \times 10^{-4}$? What is the probability that the noise value is between $-2 \times 10^{-4}$ and $10^{-4}$?

b. Given that the value of the noise is positive, what is the probability that it exceeds $10^{-4}$?

c. The noise passes through a half-wave rectifier with characteristics

\[
g(x) = \begin{cases} 
  x, & x > 0 \\
  0, & \text{otherwise}
\end{cases}
\]

Find the pdf of the rectified noise $g(X)$ by first finding its cdf.

d. Find the expected value of the rectified noise in part c.

e. Now assume that the noise passes through a full-wave rectifier defined by $g(x) = |x|$. Find the density function of the rectified noise $g(X)$.

f. What is the expected value of the output noise in part e.