Problem 1. a. Need to have \( \int_{-\infty}^{\infty} f_X(x)dx = 1 \).

\[
\int_{-\infty}^{\infty} f_X(x)dx = \int_{0}^{1} cx(1-x)dx
\]
\[
= c \int_{0}^{1} (x - x^2)dx
\]
\[
= c(\frac{x^2}{2} - \frac{x^3}{3})|_0^1
\]
\[
= c(\frac{1}{2} - \frac{1}{3})
\]
\[
= \frac{c}{6}
\]

Thus \( c = 6 \).

b.

\[
P(\frac{1}{2} \leq X \leq \frac{3}{4}) = \int_{\frac{1}{2}}^{\frac{3}{4}} f_X(x)dx
\]
\[
= \int_{\frac{1}{2}}^{\frac{3}{4}} 6x(1-x)dx
\]
\[
= 6(\frac{x^2}{2} - \frac{x^3}{3})|_{\frac{1}{2}}^{\frac{3}{4}}
\]
\[
= 6(\frac{9}{32} - \frac{9}{64} - \frac{1}{8} + \frac{1}{24})
\]
\[
= \frac{11}{32}
\]

c. The cdf is \( F_X(x) = P(X \leq x) \). For \( x < 0 \), \( F_X(x) = 0 \). For \( 0 \leq x < 1 \), \( F_X(x) = \int_0^x 6y(1-y)dy = 3x^2 - 2x^3 \). For \( x \geq 1 \), \( F_X(x) = 1 \). Hence the cdf is

\[
F_X(x) = \begin{cases} 
0, & \text{if } x < 0 \\
3x^2 - 2x^3, & \text{if } 0 \leq x < 1 \\
1, & \text{if } x \geq 1 
\end{cases}
\]
Problem 2. a. 

\[ P(X^+ \leq x) = \begin{cases} 
0, & \text{if } x < 0 \\
F(x), & \text{if } x \geq 0 
\end{cases} \]

b. 

\[ P(X^- \leq x) = \begin{cases} 
0, & \text{if } x < 0 \\
1 - \lim_{y \uparrow x} F(y), & \text{if } x \geq 0 
\end{cases} \]

c. \( P(|X| \leq x) = P(-x \leq X \leq x) \) if \( x \geq 0 \). Therefore,

\[ P(|X| \leq x) = \begin{cases} 
0, & \text{if } x < 0 \\
F(x) - \lim_{y \uparrow x} F(y), & \text{if } x \geq 0 
\end{cases} \]

d. \( P(-X \leq x) = 1 - \lim_{y \uparrow x^-} F(y) \)

Problem 3. a. i. For \( k = 1, 2, 3 \),

\[ P(X = k) = P(A_k) = \frac{\text{area}(A_k)}{9\pi} = \frac{2k - 1}{9} \]

ii. \( E[X] = 1P(X = 1) + 2P(X = 2) + 3P(X = 3) = \frac{1}{9} + \frac{2^3}{9} + \frac{3^3}{9} = \frac{22}{9} \)

b. i. 

\[ F_Y(r) = P(Y \leq r) = \begin{cases} 
0, & \text{if } r < 0 \\
\frac{r^2}{9}, & \text{if } 0 \leq r \leq 3 \\
1, & \text{if } r > 3 
\end{cases} \]
Figure 1: The distribution function $F_X$ of $X$.

Figure 2: The distribution function $F_Y$ of $Y$. 
ii. First find the pdf, \( f_Z(r) = \frac{d}{dr}F_Z(r) = \frac{2r}{9} \) for \( 0 \leq r \leq 3 \).

Now \( E[Y] = \int_0^3 r f_Z(r) dr = \int_0^3 \frac{2r^2}{9} dr = \frac{2r^3}{27} \bigg|_0^3 = 2 \).

c. i. \( P(Z \leq r) = P(Z \leq r | \text{hits target}) P(\text{hits target}) + P(Z \leq r | \text{misses target}) P(\text{misses target}) \)

Hence,

\[
F_Z(r) = P(Z \leq r) = \begin{cases} 
0, & \text{if } r < 0 \\
(1 - p) \frac{r^2}{9}, & \text{if } 0 \leq r < 3 \\
(1 - p), & \text{if } 3 \leq r < 4 \\
1, & \text{if } r \geq 4
\end{cases}
\]

ii. \( E[Z] = 2(1 - p) + 4p = 2p + 2 \)

**Problem 4.** Let \( N \) denote the number of empty boxes and \( X_i \) denote the event that the \( i \)th box is empty. Then \( P(X_i) = \left( \frac{n-1}{n^m} \right) = \left( \frac{n-1}{n} \right) \) and is the same for all \( i \).

Let \( 1(X_i) \) be the indicator function of event \( X_i \). Then \( N = \sum_{i=1}^n 1(X_i) \).

\[
E[N] = E[\sum_{i=1}^n 1(X_i)] = \sum_{i=1}^n E[1(X_i)] = \sum_{i=1}^n P(X_i) = \sum_{i=1}^n \left( \frac{n-1}{n} \right)^m = n \left( \frac{n-1}{n} \right)^m
\]
Problem 5. Let $X$ be the total number of boxes of cereal we will need to buy to collect all $n$ toys, Let $X_i$ be the number of boxes of cereal to buy to collect the $i$th distinct toy after having $i - 1$ distinct toys. Hence $X = \sum_{i=1}^{n} X_i$.

Let $P_i$ be the probability of getting the $i$th distinct toy when $i - 1$ distinct toys have already been collected. $P_i = \frac{n-i+1}{n}$ since $i - 1$ of the toys have already been chosen so there are $n - i + 1$ toys that are still distinct out of $n$. $1 - P_i$ is the probability of getting a repeat toy when $i - 1$ distinct toys have already been collected.

$$E[X_i] = \sum_{k=1}^{\infty} k(1 - P_i)^{k-1}P_i$$

$$= P_i \sum_{k=1}^{\infty} k(1 - P_i)^{k-1}$$

$$= P_i \frac{1}{[1 - (1 - P_i)]^2}$$

$$= \frac{1}{P_i}$$

$$= \frac{n}{n - i + 1}$$

$$E[X] = E[\sum_{i=1}^{n} X_i]$$

$$= \sum_{i=1}^{n} E[X_i]$$

$$= \sum_{i=1}^{n} \frac{n}{n - i + 1}$$

$$= n\left(\sum_{i=1}^{n} \frac{1}{i}\right)$$

Hence the expected amount of money one needs to spend to collect all $n$ toys is

$$C \times n\sum_{i=1}^{n} \frac{1}{i}.$$  

Problem 6. Since $P$ is chosen uniformly on the square, the probability we are within some region of the square is just proportional to the area of that region.

First find the cdf.

$$F_\Theta(\theta) = P(\Theta \leq \theta) = \begin{cases} 
0, & \text{if } \theta < 0 \\
\frac{1}{2} \tan \theta, & \text{if } 0 \leq \theta \leq \frac{\pi}{4} \\
1 - \frac{1}{2} \tan \left(\frac{\pi}{2} - \theta\right), & \text{if } \frac{\pi}{4} < \theta \leq \frac{\pi}{2} \\
1, & \text{if } \theta > \frac{\pi}{2}
\end{cases}$$
Then we can differentiate the cdf to find the pdf.

\[ f_\Theta(\theta) = \frac{d}{d\theta} F_\Theta(\theta) = \begin{cases} 
\frac{1}{2(\cos \theta)^2}, & \text{if } 0 \leq \theta \leq \frac{\pi}{4} \\
\frac{1}{2[\cos(\frac{\pi}{2} - \theta)]^2}, & \text{if } \frac{\pi}{4} < \theta \leq \frac{\pi}{2} \\
0, & \text{otherwise}
\end{cases} \]

Use the pdf to find the expected value.

\[ E[\Theta] = \int_0^{\frac{\pi}{4}} \frac{\theta}{2(\cos \theta)^2} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\theta}{2[\cos(\frac{\pi}{2} - \theta)]^2} d\theta 
= \frac{1}{2} [\ln(\cos \theta) + \theta \tan \theta]_0^{\frac{\pi}{4}} + \frac{1}{2} [\ln(\cos(\frac{\pi}{2} - \theta)) - \theta \tan(\frac{\pi}{2} - \theta)]_{\frac{\pi}{4}}^{\frac{\pi}{2}} 
= \frac{1}{2} [\ln(\frac{\sqrt{2}}{2}) + \frac{\pi}{4} - (\ln(\frac{\sqrt{2}}{2}) - \frac{\pi}{4})] 
= \frac{\pi}{4} \]

**Problem 7.** Let \( X \) denote the noise voltage in an electric circuit. \( X \) is modeled as a Gaussian random variable with mean equal to zero and variance equal to \( 10^{-8} \), so \( f_X(x) = \frac{1}{\sqrt{2\pi}10^{-4}} \exp\left[-\frac{x^2}{2 \times 10^{-8}}\right] \).

a. Recall from discussion that when \( X \) is Gaussian with mean zero and variance \( \sigma^2 \), then \( P(X > x) = Q(\frac{x}{\sigma}) \).

\[ P(X > 10^{-4}) = \int_{10^{-4}}^{\infty} \frac{1}{\sqrt{2\pi}10^{-4}} \exp\left[-\frac{x^2}{2 \times 10^{-8}}\right] dx 
= \int_1^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u^2}{2}\right] du 
= Q(1) 
= 0.159 \]

\[ P(X > 4 \times 10^{-4}) = \int_4^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u^2}{2}\right] du = Q(4) = 3.17 \times 10^{-5} \]

\[ P(-2 \times 10^{-4} < x \leq 10^{-4}) = 1 - Q(1) - Q(2) = 0.8182 \]

b. \( P(X > 10^{-4}|X > 0) = \frac{P(X > 10^{-4} \cap X > 0)}{P(X > 0)} = \frac{P(X > 10^{-4})}{P(X > 0)} = \frac{Q(1)}{Q(0)} = 0.318 \)

c. Let \( Y = g(X) \). We first find the cdf of \( Y \).

\[ F_Y(y) = \begin{cases} 
P(X \leq y), & \text{if } y \geq 0 \\
0, & \text{if } y < 0
\end{cases} = \begin{cases} 
\frac{1}{2} + \int_0^y f_X(x) dx, & \text{if } y \geq 0 \\
0, & \text{if } y < 0
\end{cases} \]
Notice that $F_Y(y)$ is discontinuous at $y = 0$ and that discontinuity equals $F_X(0)$ since cdf’s need to be right continuous.

Then the pdf is

$$f_Y(y) = \begin{cases} 
  f_X(y), & \text{if } y > 0 \\
  \frac{1}{2}, & \text{if } y = 0 \\
  0, & \text{if } y < 0 
\end{cases}$$

Succinctly, $f_Y(y) = f_X(y)u(y) + \frac{\delta(y)}{2}$ where $u(\cdot)$ is the unit step function and $\delta(\cdot)$ is the delta function.

d. 

$$E[g(X)] = E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \frac{1}{10^{-4} \sqrt{2\pi}} \int_{0}^{\infty} \exp\left[-\frac{y^2}{2 \times 10^{-8}}\right] dy = \frac{10^{-4}}{\sqrt{2\pi}}$$

e. Let $Z = g(X)$ where $g(x) = |x|$. Find the density function of the rectified noise $Z$ by first finding its cdf.

$$F_Z(z) = P(Z \leq z) = P(|X| \leq z) = P(-z \leq X \leq z) = \int_{-z}^{z} f_X(x) dx = 2 \int_{0}^{z} f_X(x) dx$$

Since $f_X(\cdot)$ is symmetric about 0.

$$f_Z(z) = \begin{cases} 
  2f_X(z), & \text{if } y \geq 0 \\
  0, & \text{if } y < 0 
\end{cases} = \begin{cases} 
  \frac{1}{(2 \times 10^{-4}) \sqrt{2\pi}} \exp\left[-\frac{x^2}{2 \times 10^{-8}}\right], & \text{if } y \geq 0 \\
  0, & \text{if } y < 0 
\end{cases}$$

f. 

$$E[g(X)] = E[Z]$$
\[ = \int_0^\infty z f_Z(z) dz \]

\[ = \frac{1}{(2 \times 10^{-4})\sqrt{2\pi}} \int_0^\infty z \exp\left[ -\frac{z^2}{2 \times 10^{-8}} \right] dz \]

\[ = \frac{2 \times 10^{-4}}{\sqrt{2\pi}} \]