Problem 1. For this problem we need to construct a pdf for which \( P[X > a + b] > P[X > a]P[X > b] \).

Consider the distribution, \( f_X(x) = \frac{1}{(x+1)^2}, 0 \leq x < \infty \).

First we need to check that \( f_X(x) \) is indeed a pdf:

\[
\int_0^\infty \frac{1}{(x+1)^2} dx = \left[ -\frac{1}{x+1} \right]_0^\infty = 1
\]

\[
P[X > h] = 1 - F_X(h) = \frac{1}{h+1}
\]

\[
P[X > a + b] = 1 - F_X(a + b) = \frac{1}{a+b+1}
\]

\[
P(X > a)P(X > b) = \frac{1}{(a+1)(b+1)}
\]

Hence,

\[
\frac{P(X>a+b)}{P(X>a)P(X>b)} = \frac{1+ab+a+b}{a+b+1}
\]

which is strictly greater than 1 since \( a > 0, b > 0 \).

Hence \( P[X > a + b] > P[X > a]P[X > b] \).

Problem 2. For this problem we need to construct a pdf for which \( P[X > a + b] < P[X > a]P[X > b] \).

For the exponential distribution we have \( P[X > a + b] = P[X > a]P[X > b] \).

So we want to check variants of the exponential distribution and get the distribution we want.

Consider the Raleigh distribution, \( f_X(x) = xe^{-x^2/2\alpha^2}, 0 \leq x < \infty \).

\[
P[X > h] = 1 - F_X(h) = e^{-h^2/2\alpha^2}
\]

\[
P[X > a + b] = 1 - F_X(a + b) = e^{-(a+b)^2/2\alpha^2}
\]

\[
P(X > a)P(X > b) = (1 - F_X(a))(1 - F_X(b)) = e^{-(a^2+b^2)/2\alpha^2}
\]
Hence,
\[ \frac{P(X > a + b)}{P(X > a)P(X > b)} = \frac{e^{(2ab)}/2\alpha^2}{2\alpha^2} \]
which is strictly less than 1 since \( a > 0, b > 0 \)
Hence \( P[X > a + b] < P[X > a]P[X > b] \)

**Problem 3.** It’s important to remember that,
\[
f_{XY}(x, y) = \begin{cases} 
2 & 0 \leq x, y \leq 1, y < x \\
0 & \text{otherwise}
\end{cases}
\]

Also \( \int_0^1 \int_0^x f_{XY}(x, y) \, dx \, dy = 1 \)
The marginal pdf of \( Y \) is given by \( f_Y(y) = \int_y^1 f_{XY}(x, y) \, dx = 2(1 - y) \)

a.
When we choose a particular value of \( Y \), say \( y \) then we know that \( X \) will lie between \( y \) and 1. Hence, \( f_{X|Y}(x|y) = \frac{1}{y - 1} \).
\( E[X|Y = y] = \int_y^1 x f_{X|Y}(x|y) \, dx = \frac{1 + y}{2} \)
Hence \( E[X|Y = y] = \frac{1 + y}{2} \).
Remember that that \( E[X|Y] \) is a Random variable and a function of \( Y \).
Hence we can write \( E[X|Y] = \frac{1 + Y}{2} \)

b. Similarly, when we choose a particular value of \( X \), say \( x \) then we know that \( Y \) will lie between 0 and \( x \). Hence, \( f_{Y|X}(y|x) = \frac{1}{x} \).
\( E[Y|X = x] = \int_0^x y f_{Y|X}(x|y) \, dy = \frac{x}{2} \)
\( E[Y|X = y] = \frac{y}{2} \).
Hence \( E[Y|X] = \frac{X}{2} \)

c.
First let’s calculate, \( E[Y^2|X = y] = \int_0^x y^2 f_{Y|X}(y|x) \, dy = \frac{y^2}{3} \)
Hence \( E[Y^2|X] = \frac{X^2}{3} \)

\[
E[(X - Y)^2|X] = E[(X^2 - 2XY + Y^2)|X] = E[X^2|X] + E[2XY|X] + E[Y^2|X] = X^2 + 2XE[Y|X] + \frac{X^2}{3} = X^2 + X^2 + \frac{X^2}{3} = \frac{7X^2}{3}
\]
Problem 4. a. We know that, $E[X|Y]$ is that function of $y$ that minimizes the Mean Square Error i.e. $E[(X - E[X|Y])^2] < E[(X - g(Y))^2]$. Hence we need to construct $g(Y)$ so that it is closest to $E[X|Y]$ in the Mean Square sense. In Figure 1 we can see the plot of $E[X|Y]$ and the function $g(Y)$. Since $E[X|Y]$ takes on values between 0.5 and 1, $g(Y)$ should take on values 0.5 and 1. Hence $g(Y)$ is a function of the form:

$$g(y) = \begin{cases} 
0.5 & 0 \leq y \leq a \\
1 & a < y \leq 1 
\end{cases}$$

Next we have to estimate the value of $a$ for which $E[(g(Y) - E[X|Y])^2]$ is minimized.

$$E[(g(Y) - E[X|Y])^2] = \int_0^a \left( \frac{y + 1}{2} - \frac{1}{2} \right)^2 f_Y(y)dy + \int_a^1 \left( \frac{y + 1}{2} - 1 \right)^2 f_Y(y)dy$$

$$= \int_0^a \frac{(y - 1)^2}{2} + 2(1 - y)dy + \int_a^1 \frac{(y - 1)^2}{2} + 2(1 - y)dy$$

We need to minimize $E[(g(Y) - E[X|Y])^2]$. Taking the derivative of $E[(g(Y) - E[X|Y])^2]$ wrt $a$. We get
\[\frac{a}{2}^2 \cdot 2(1-a) - (\frac{a-1}{2})^2 \cdot 2(1-a) = 0 \]
\[\frac{a}{2}^2 \cdot 2(1-a) = (\frac{a-1}{2})^2 \cdot 2(1-a) \]
\[2a - 1 = 0 \]
\[a = 0.5 \]

Hence,
\[g(y) = \begin{cases} 0.5 & 0 \leq y \leq 0.5 \\ 1 & 0.5 < y \leq 1 \end{cases} \]

b. When \(Y\) takes on the value \(y\), we know that \(X\) lies between \(y\) and 1. Let say that \(g(Y)\) takes on a value of \(a\) when \(Y = y\). Clearly \(y \leq a \leq 1\).

\[E[|X - g(Y)||Y = y] = \int_y^a |x - a| f_X|Y(x|y)dx + \int_a^1 |x - a| f_X|Y(x|y)dx \]
\[= \int_y^a (a - x) \frac{1}{1-y} dx + \int_a^1 (x - a) \frac{1}{y-1} dx \]
\[= \frac{1}{1-y} \left( (ax - \frac{x^2}{2})|_y + \frac{x^2}{2} - ax \right) \]
\[= \frac{1}{1-y} \left( \frac{a^2}{2} - ay + \frac{y^2}{2} + \frac{1}{2} - a + \frac{a^2}{2} \right) \]
\[= \frac{1}{1-y} \left( a^2 - ay - \frac{y^2}{2} + \frac{1}{2} - a \right) \]

Next differentiating \(E[|X - g(Y)||Y = y]\) wrt \(a\) and equating to 0 we get \(a = \frac{y+1}{2}\).

Hence \(g(Y) = \frac{Y+1}{2}\).

Problem 5a. \(Y\) takes on three values, 1, 2 and 3.

When \(Y = 0\), \(8 < X \leq 10\) and uniformly distributed between 8 and 10 hence \(E[X|Y = 0] = 9\).

Similarly \(E[X|Y = 1] = 7\), \(E[X|Y = 2] = 5.5\) and \(E[X|Y = 3] = 5\).

Hence \(E[X|Y] = 9 \cdot 1(Y = 0) + 6 \cdot 1(Y = 1) + 5.5 \cdot 1(Y = 2) + 5 \cdot 1(Y = 3)\)

b. \(g(Y)\) is of the form,

\[g(Y) = \begin{cases} 
  g_0 & Y = 0 \\
  g_1 & Y = 1 \\
  g_2 & Y = 2 \\
  g_3 & Y = 3 
\end{cases} \]
\[ E[(|X - g(Y)|)|Y = 0] = \int_8^{g_0} |x - g_0| f_{X|Y}(x|y) dx + \int_{g_0}^{10} |x - a| f_{X|Y}(x|y) dx \]
\[ = \int_8^{g_0} \frac{1}{2} (g_0 - x) dx + \int_{g_0}^{10} \frac{1}{2} (x - g_0) dx \]
\[ = \frac{1}{2} \left[ (g_0 x - \frac{x^2}{2})_{g_0}^8 + \left( \frac{x^2}{2} - g_0 x \right)_{g_0}^{10} \right] \]
\[ = \frac{1}{2} \left( \frac{g_0^2}{2} - 8g_0 + 32 + 50 - 10g_0 + \frac{g_0^2}{2} \right) \]
\[ = \frac{1}{2} (g_0^2 - 18g_0 + 82) \]

Next differentiating \( E[(|X - g(Y)|)|Y = 1] \) wrt \( g_0 \) and equating to 0 we get \( g_0 = 9 \).

Similarly, we can obtain \( E[(|X - g(Y)|)|Y = 1] = 7 \), \( E[(|X - g(Y)|)|Y = 2] = 5.5 \) and \( E[(|X - g(Y)|)|Y = 3] = 5 \)

Hence,

\[ g(Y) = 9 1(Y = 0) + 7 1(Y = 1) + 5.5 1(Y = 2) + 5 1(Y = 3) \]

c.

As seen in (b) above, \( g(Y) \) is of the form,

\[ g(Y) = \begin{cases} 
g_0 & Y = 0 
g_1 & Y = 1 
g_2 & Y = 2 
g_3 & Y = 3 
\end{cases} \]

\[ E[(X - g(Y))^4|Y = 0] = \int_8^{10} (x - g_0)^4 f_{X|Y}(x|y) dx \]
\[ = \int_8^{10} \frac{1}{2} (x - g_0)^4 dx \]
\[ = \frac{1}{2} \left( \frac{(x - g_0)^5}{5} \right)_{8}^{10} \]
\[ = \frac{(10 - g_0)^5 - (8 - g_0)^5}{10} \]

Next differentiating \( E[(X - g(Y))^4|Y = 0] \) wrt \( g_0 \) and equating to 0 we get \( g_0 = 9 \).

Similarly, we can obtain \( E[(X - g(Y))^4|Y = 1] = 7 \), \( E[(X - g(Y))^4|Y = 2] = 5.5 \) and \( E[(X - g(Y))^4|Y = 3] = 5 \)

Hence,

\[ g(Y) = 9 1(Y = 0) + 7 1(Y = 1) + 5.5 1(Y = 2) + 5 1(Y = 3) \]
Problem 6 We know that $E[X|Y]$ satisfies the equation $E[(X - E[X|Y])g(Y)]$

Hence taking $g(Y) = Y$, we have $E[(X - E[X|Y])Y] = 0$ or $E[(X - Y)Y] = 0$

Similarly we have, $E[(Y - X)X] = 0$


Let $W = X - Y$. So far we have shown that $E[W^2] = 0$. We need to show that $E[W^2] = 0 \Rightarrow P(W = 0) = 1$.

Method 1: We can prove this using Markov’s inequality:

$$P(W \geq 0+) \leq P(|W| \geq 0+) \leq \frac{E[W^2]}{(0+)^2} = 0$$

Hence $P(W > 0) = 0$ and $P(W = 0) = 1$

Method 2: Let assume that $P(W = 0) = \alpha$ where $\alpha < 1$ and $E[W^2] = 0$. If $P(W = 0) < 1$ then there exists some $\beta \neq 0$ such that $P(W = \beta) = 1 - \alpha$

Hence $E[W^2] = 0 \times \alpha + \beta^2 \times (1 - \alpha) = \beta^2 \times (1 - \alpha)$.

Since $\beta \neq 0$ and $1 - \alpha > 0$, $E[W^2] > 0$ which leads to a contradiction.

Hence $P((X - Y) = 0) = 1$ is true

Problem 7 Let’s consider three cases:

Case I: $m = n$

$$E[X_1 + \cdots + X_n | X_1 + \cdots + X_n] = X_1 + \cdots + X_n$$

Case II: $m > n$

Remember that $X_p, p > n$, is independent of $X_q, q \leq n$ $E[X_1 + \cdots + X_n + X_{n+1} \cdots + X_m | X_1 + \cdots + X_m] =

= E[X_1 + \cdots + X_n | X_1 + \cdots + X_n] + E[X_{n+1} + \cdots + X_m | X_1 + \cdots + X_n]$

= $X_1 + \cdots + X_n + E[X_{n+1} | X_1 + \cdots + X_n] + \cdots E[X_m | X_1 + \cdots + X_n]$

= $X_1 + \cdots + X_n + X_{n+1} + \cdots + X_m$

Case III: $m < n$

$$E[X_1 + \cdots + X_m | X_1 + \cdots + X_n] =

= E[X_1 + \cdots + X_n | X_1 + \cdots + X_n] - E[X_{m+1} + \cdots + X_m | X_1 + \cdots + X_n]$

= $X_1 + \cdots + X_n - E[X_{m+1} | X_1 + \cdots + X_n] \cdots - E[X_n | X_1 + \cdots + X_n]$

By symmetry we see that $E[X_i | X_1 + \cdots + X_n], (m + 1) \leq i \leq n$ are all the same, so we can calculate one of them. Let’s calculate, $E[X_n | X_1 + \cdots + X_n]$. 

6
\[ P(X_n = a | X_1 + \cdots + X_n = b) = \frac{P(X_n=a, X_1 + \cdots + X_n = b)}{P(X_1 + \cdots + X_n = b)} \]

It's important to remember that the sum of \( n \) independent Poisson Random Variables with parameter \( \lambda \) is a Poisson Random with parameter \( n\lambda \)

If \( b \geq a \),

\[ P(X_n = a, X_1 + \cdots + X_n = b) = P(X_1 + \cdots + X_{n-1} = b-a, X_n = a) \]

\[ = P(X_1 + \cdots + X_{n-1} = b-a)P(X_n = a) \]

\[ = \frac{((n-1)\lambda)^{b-a}e^{(n-1)\lambda} (\lambda)^ae^\lambda}{(b-a)!} \]

Hence \( \frac{P(X_n=a, X_1 + \cdots + X_n = b)}{P(X_1 + \cdots + X_n = b)} = \frac{b}{a!} \left( \frac{1}{n} \right)^a (\frac{n-1}{n})^{b-a} \)

The above is the probability of a Binomial Random Variable with parameters \( B(b, \frac{1}{n}) \).

Hence \( E[X_n | X_1 + \cdots + X_n = b] = \sum_{a=0}^{b} aP(X_n = a | X_1 + \cdots + X_n = b) = \frac{b}{n} \)

and \( E[X_n | X_1 + \cdots + X_n] = \frac{X_1 + \cdots + X_n}{n} \)

Hence \( E[X_1 + \cdots + X_m + X_{m+1} + \cdots + X_n | X_1 + \cdots + X_n] = X_1 + \cdots + X_n - \frac{n-m}{n}(X_1 + \cdots + X_n) = \frac{m}{n}(X_1 + \cdots + X_n) \)

An easier way to see the same, is to go back to Case I. If we take \( E[X_i | X_1 + \cdots + X_n] = \alpha, (0 \leq i \leq n) \) then we have \( n\alpha = X_1 + \cdots + X_n \)

Hence \( E[X_1 + \cdots + X_m | X_1 + \cdots + X_n] = \alpha \frac{m}{n} = \frac{ma}{n}(X_1 + \cdots + X_n) \)