Problem 1. Let $X, Y, Z, V$ be independent $N(0, 1)$ random variables. Find the p.d.f. of $X^2 + Y^2 + Z^2 + V^2$.

This problem is definitely tricky and hard to do without checking reference texts unless you know the first step.

First, we claim that $W := X^2 + Y^2 = d \text{ Ex}(1/2)$. To see this we calculate the characteristic function of $W$. We find

$$E(e^{iuW}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iu(x^2+y^2)} \frac{1}{2\pi} e^{-(x^2+y^2)/2} dxdy$$

$$= \int_{0}^{2\pi} \int_{0}^{\infty} e^{iur} \frac{1}{2\pi} e^{-r^2/2} rdrd\theta$$

$$= \int_{0}^{\infty} e^{iur} e^{-r^2/2} rdr$$

$$= \int_{0}^{\infty} \frac{1}{2iu-1} d[e^{iur^2-r^2/2}] = \frac{1}{1-2iu}.$$ 

On the other hand, if $W = d \text{ Ex}(\lambda)$, then

$$E(e^{iuW}) = \int_{0}^{\infty} e^{iux} \lambda e^{-\lambda x} dx$$

$$= \frac{\lambda}{\lambda - iu} = \frac{1}{1 - \lambda^{-1}iu}.$$

Comparing these expressions shows that $X^2 + Y^2 = d \text{ Ex}(1/2)$ as claimed.

Second, we note that $U := X^2 + Y^2 + Z^2 + V^2$ is the sum of two independent $\text{Ex}(1/2)$ random variables. This sum is not exponentially distributed. We can find its density as the convolution of the densities of the two $\text{Ex}(1/2)$ random variables. That is, with $\lambda = 1/2$,

$$f_U(u) = \int_{0}^{u} \lambda e^{-\lambda x} \lambda e^{-\lambda(u-x)} dx = \lambda^2 u e^{-\lambda u}.$$
**Problem 2.** Let \( \{X_n, n \geq 0\} \) be Gaussian \( N(0, 1) \) random variables. Assume that \( Y_{n+1} = aY_n + X_n \) for \( n \geq 0 \) where \( Y_0 \) is a Gaussian random variable with mean zero and variance \( \sigma^2 \) independent of the \( X_n \)'s and \( |a| < 1 \).

a. Calculate \( \text{var}(Y_n) \) for \( n \geq 0 \). Show that \( \text{var}(Y_n) \to \gamma^2 \) as \( n \to \infty \) for some value \( \gamma^2 \).

b. Find the values of \( \sigma^2 \) so that the variance of \( Y_n \) does not dependent on \( n \geq 1 \).

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a. We see that

\[
\text{var}(Y_{n+1}) = \text{var}(aY_n + X_n) = a^2\text{var}(Y_n) + \text{var}(X_n) = a^2\text{var}(Y_n) + 1.
\]

Thus, we define \( \alpha_n := \text{var}(Y_n) \), one has

\[
\alpha_{n+1} = a^2\alpha_n + 1 \text{ and } \alpha_0 = \sigma^2.
\]

Solving these equations we find

\[
\text{var}(Y_n) = \alpha_n = a^{2n}\sigma^2 + \frac{1 - a^{2n}}{1 - a^2}, \text{ for } n \geq 0.
\]

Since \( |a| < 1 \), it follows that

\[
\text{var}(Y_n) \to \gamma^2 := \frac{1}{1 - a^2} \text{ as } n \to \infty.
\]

b. The obvious answer is \( \sigma^2 = \gamma^2 \).
**Problem 3.** Let the $X_n$’s be as in Problem 2.

a. Calculate
\[ E[X_1 + X_2 + X_3 | X_1 + X_2, X_2 + X_3, X_3 + X_4]. \]

b. Calculate
\[ E[X_1 + X_2 + X_3 | X_1 + X_2 + X_3 + X_4 + X_5]. \]

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a. We know that the solution is of the form $Y = a(X_1 + X_2) + b(X_2 + X_3) + c(X_3 + X_4)$ where the coefficients $a, b, c$ must be such that the estimation error is orthogonal to the conditioning variables. That is,
\[ E((X_1 + X_2 + X_3) - Y)(X_1 + X_2)) = E((X_1 + X_2 + X_3) - Y)(X_2 + X_3)) = E((X_1 + X_2 + X_3) - Y)(X_3 + X_4)) = 0. \]

These equalities read
\[ 2 - a - (a + b) = 2 - (a + b) - (b + c) = 1 - (b + c) - c = 0, \]
and solving these equalities gives $a = 3/4, b = 1/2, \text{ and } c = 1/4.$

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b. Here we use symmetry. For $k = 1, \ldots, 5$, let
\[ Y_k = E[X_k | X_1 + X_2 + X_3 + X_4 + X_5]. \]
Note that $Y_1 = Y_2 = \cdots = Y_5,$ by symmetry. Moreover,
\[ Y_1 + Y_2 + Y_3 + Y_4 + Y_5 = E[X_1 + X_2 + X_3 + X_4 + X_5 | X_1 + X_2 + X_3 + X_4 + X_5] = X_1 + X_2 + X_3 + X_4 + X_5. \]

It follows that $Y_k = (X_1 + X_2 + X_3 + X_4 + X_5)/5$ for $k = 1, \ldots, 5.$ Hence,
\[ E[X_1 + X_2 + X_3 | X_1 + X_2 + X_3 + X_4 + X_5] = Y_1 + Y_2 + Y_3 = \frac{3}{5}(X_1 + X_2 + X_3 + X_4 + X_5). \]
**Problem 4.** Let the $X_n$'s be as in Problem 2. Find the j.p.d.f. of $(X_1 + 2X_2 + 3X_3, 2X_1 + 3X_2 + X_3, 3X_1 + X_2 + 2X_3)$.

These random variables are jointly Gaussian, zero mean, and with covariance matrix $\Sigma$ given by

$$
\Sigma = \begin{bmatrix}
14 & 11 & 11 \\
11 & 14 & 11 \\
11 & 11 & 14
\end{bmatrix}.
$$

Indeed, $\Sigma$ is the matrix of covariances. For instance, its entry (2, 3) is given by

$$
E((2X_1 + 3X_2 + X_3)(3X_1 + X_2 + 2X_3)) = 2 \times 3 + 3 \times 1 + 1 \times 2 = 11.
$$

We conclude that the j.p.d.f. is

$$
f_X(x) = \frac{1}{(2\pi)^{3/2}|\Sigma|^{1/2}} \exp\left\{ -\frac{1}{2}x^T\Sigma^{-1}x \right\}.
$$

We could calculate $|\Sigma|$ and $\Sigma^{-1}$ ....

**Problem 5.** Let $X_1, X_2, X_3$ be independent $N(0, 1)$ random variables. Calculate $E[X_1 + 3X_2 | Y]$ where

$$
Y = \begin{bmatrix}
1 & 2 & 3 \\
3 & 2 & 1
\end{bmatrix} \begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix}
$$

By now, this is familiar stuff... The solution is $Y := a(X_1 + 2X_2 + 3X_3) + b(3X_1 + 2X_2 + X_3)$ where $a$ and $b$ are such that

0 = $E((X_1 + 3X_2 - Y)(X_1 + 2X_2 + 3X_3)) = 7 - (a + 3b) - (4a + 4b) - (9a + 3b) = 7 - 14a - 10b$

and

0 = $E((X_1 + 3X_2 - Y)(3X_1 + 2X_2 + X_3)) = 9 - (3a + 9b) - (4a + 4b) - (3a + b) = 9 - 10a - 14b$.

Solving these equations gives $a = 1/12$ and $b = 7/12$. 

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**Problem 6.** Find the j.p.d.f. of $(2X_1 + X_2, X_1 + 3X_2)$ where $X_1$ and $X_2$ are independent $N(0, 1)$ random variables.

These random variables are jointly Gaussian, zero-mean, with covariance $\Sigma$ given by

$$
\Sigma = \begin{bmatrix}
5 & 5 \\
5 & 10
\end{bmatrix}.
$$

Hence,

$$
f_X(x) = \frac{1}{2\pi|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}x^T \Sigma^{-1} x\right\}
= \frac{1}{10\pi} \exp\left\{-\frac{1}{2}x^T \Sigma^{-1} x\right\}
$$

where

$$
\Sigma^{-1} = \frac{1}{25} \begin{bmatrix}
10 & -5 \\
-5 & 5
\end{bmatrix}.
$$

**Problem 7.** The random variable $X$ is $N(\mu, 1)$. Find an approximate value of $\mu$ so that

$$
P(-0.5 \leq X \leq -0.1) \approx P(1 \leq X \leq 2).
$$

We write $X = \mu + Y$ where $Y$ is $N(0, 1)$. We must find $\mu$ so that

$$
g(\mu) := P(-0.5 - \mu \leq Y \leq -0.1 - \mu) - P(1 - \mu \leq Y \leq 2 - \mu) \approx 0.
$$

We do a little search using a table of the $N(0, 1)$ distribution or using a calculator. I find that $\mu \approx 0.065$. 

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Problem 8. Let $X$ be a $N(0, 1)$ random variable. Calculate the mean and the variance of $\cos(X)$ and $\sin(X)$.

We know that

$$E(e^{iuX}) = e^{-u^2/2} \text{ and } e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Therefore,

$$E(\cos(uX) + i\sin(uX)) = e^{-u^2/2},$$

so that

$$E(\cos(uX)) = e^{-u^2/2} \text{ and } E(\sin(uX)) = 0.$$

In particular, $E(\cos(X)) = e^{-1/2}$ and $E(\sin(X)) = 0.$

Problem 9. Let $X$ be a $N(0, 1)$ random variable. Define

$$Y = \begin{cases} X, & \text{if } |X| \leq 1 \\ -X, & \text{if } |X| > 1. \end{cases}$$

Find the p.d.f. of $Y$.

By symmetry, $X$ is $N(0, 1).$