Problem 1. We have gone through this problem in the discussions. Here are the answers.

a. \( \hat{X} = 1(1Y = 1) + 0(1Y = 0) \)

b. \( \hat{X} = (1)1(Y = 0, p > 0.7) + (0)1(Y = 0, p \leq 0.7) + (1)1(Y = 1, p > \frac{1}{3}) + (0)1(Y = 1, p \leq \frac{1}{3}) \)

c. Case I: 0 \leq \beta \leq 0.3

\[ \lambda = 2 \]
\[ \gamma = \frac{\beta}{0.3} \]
\( \hat{X} = 1 \) with probability \( \gamma \) when \( Y = 1 \)

Case II: 0.3 < \beta \leq 1

\[ \lambda = \frac{4}{7} \]
\[ \gamma = \frac{\beta - 0.3}{0.7} \]
\( \hat{X} = 1(1Y = 1) + 1 \) with probability \( \gamma \) when \( Y = 0 \)

Problem 2a. Recall that the MLE estimate is the decision that chooses that value of \( X \) that maximizes the probability of the observation \( Y \).

Let \( \hat{X} \) denote the decided value of \( X \). Then,
\( \hat{X} = 1 \) if,

\[ f_{Y|X=1}(Y|X = 1) > f_{Y|X=2}(Y|X = 2) \]

\[ e^{-\sum_{i=1}^{n}Y_i} > \left(\frac{1}{2}\right)^{n}e^{-\frac{1}{2}\sum_{i=1}^{n}y_i} \]

\[ e^{\frac{1}{2}\sum_{i=1}^{n}Y_i} < 2^{n} \]

\[ \sum_{i=1}^{n}Y_i < 2 \ln(2^{n}) \]

Hence \( \hat{X} = (1)1(\sum_{i=1}^{n}Y_i < 2 \ln(2^{n})) + (2)1(\sum_{i=1}^{n}Y_i > 2 \ln(2^{n})) \)
b. The MAP estimate is the decision that chooses that value of \( X \) that is most likely given the observation.

\[
\hat{X} = 1 \text{ if,} \quad f_{X=1|Y}(X = 1|Y) > f_{X=2|Y}(X = 2|Y)
\]

\[
P(X = 1)f_{Y|X=1}(Y|X = 1) > P(X = 2)f_{Y|X=2}(Y|X = 2)
\]

\[
p e^{-\sum_{i=1}^{n} Y_i} > (1 - p)(\frac{1}{2})^{n} e^{-\frac{1}{2} \sum_{i=1}^{n} Y_i}
\]

\[
e^{\frac{1}{2} \sum_{i=1}^{n} Y_i} < \frac{p}{1 - p} 2^n
\]

\[
\sum_{i=1}^{n} Y_i < 2 \,(ln(2^n) + ln(\frac{p}{1 - p}))
\]

Hence \( \hat{X} = (1) \, 1(\sum_{i=1}^{n} Y_i < 2 \,(ln(2^n) + ln(\frac{p}{1 - p})) + (2) \, 1(\sum_{i=1}^{n} Y_i > 2 \,(ln(2^n) + ln(\frac{p}{1 - p}))) \)

c. Let \( Y = (Y_1, Y_2, \cdots , Y_n) \). First let us compute the likelihood ratio \( L(Y) \).

\[
L(Y) = \frac{f_{Y|X=2}(Y|X = 2)}{f_{Y|X=1}(Y|X = 1)}
\]

We seek the threshold \( \lambda \) such that we decide \( \hat{X} = 2 \) if \( L(Y) > \lambda \)

\[
\left(\frac{1}{2}\right)^{n} e^{\frac{1}{2} \sum_{i=1}^{n} Y_i} > \lambda
\]

\[
\sum_{i=1}^{n} Y_i > 2ln(2^n \lambda)
\]

Let us denote the random variable \( S \) as \( \sum_{i=1}^{n} Y_i \). Since \( S \) is the sum of \( n \) exponential random variables it will have an m-erlang distribution.

The decision rule can be stated more conveniently as: \( \hat{X}_n = 1(S > \lambda_n) \)

where, \( \lambda_n = 2 \, ln(2^n \lambda) \)

The probability of false alarm is given by

\[
P(\hat{X}_n = 2|X = 1) = \int_{\lambda_n}^{\infty} \frac{s^{n-1} e^{-s}}{(n - 1)!} \, ds
\]

\[
= 1 - F_{S|X=1}(\lambda_n|X = 1)
\]

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where, $F_S(s)$ is the cdf of the m-erlang distribution. Unfortunately $F_S(s)$ has no closed form solution.
d. Similarly,

$$P(\hat{X}_n = 1|X = 2) = \int_0^{\lambda_n} \frac{\left(\frac{1}{2}\right)^{n-1} s^{n-1} e^{-\frac{1}{2}s}}{(n-1)!} ds$$

$$= F_{S|X=2}(\lambda_n|X = 2)$$
e.

**Problem 3.** Define $Z = X + Y$. Then $f_{Z|X=x} = N(x, \sigma^2)$

The estimated value of $X$, $(g(Z) = \hat{X})$ can be expressed as

$g(Z) = \hat{X} = (-1) 1(Z < b) + (0) 1(b \leq Z \leq a) + (1) 1(Z > a)$

Then,

$$P(X \neq \hat{X}) = P(X = 1, Z < a) + P(X = -1, Z > b) + P(X = 0, Z > a) + P(X = 0, Z < b)$$

Now $P(X = 1, Z < a) = P(X = 1)F_{Z\leq a|X=1}(a|X = 1) = \frac{1}{3} (1 - Q(\frac{a-1}{\sigma}))$

Hence,

$$P(X \neq \hat{X}) = \frac{1}{3} [(1 - Q(\frac{a-1}{\sigma})) + Q(\frac{b+1}{\sigma}) + Q(\frac{a}{\sigma}) + 1 - Q(\frac{b}{\sigma})]$$

To minimize $P(X \neq \hat{X})$ we need to differentiate the above expression wrt $a$ and $b$ and equate the partial derivatives to 0 i.e.

$$\frac{\partial}{\partial a} P(X \neq \hat{X}) = 0$$

$$\frac{\partial}{\partial b} P(X \neq \hat{X}) = 0$$

$$\frac{\partial}{\partial a} P(X \neq \hat{X}) = 0$$

$$\frac{\partial}{\partial b} P(X \neq \hat{X}) = 0$$

Similarly,

$$\frac{\partial}{\partial a} P(X \neq \hat{X}) = 0$$

$$\frac{\partial}{\partial b} P(X \neq \hat{X}) = 0$$
\[
\frac{1}{3} \left( -\frac{1}{\sqrt{2\pi}} e^{-\frac{(b+1)^2}{2\sigma^2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{b^2}{\sigma^2}} \right) = 0
\]
\[
e^{-\frac{1}{2}\frac{(b+1)^2}{2\sigma^2}} = e^{-\frac{1}{2}\frac{b^2}{2\sigma^2}}
\]
\[
(b + 1)^2 = b^2
\]
\[
b = -\frac{1}{2}
\]

Hence,
\[
g(Z) = \hat{X} = (-1) 1(\frac{-1}{2} < Z < \frac{-1}{2}) + (0) 1(\frac{-1}{2} \leq Z \leq \frac{1}{2}) + (1) 1(Z > \frac{1}{2})
\]

**Problem 4a.** \(Y\) is a column vector consisting of \((Y_1, Y_2)\).

Let \(Y = Z + X\), where \(Z = N(0, I)\)

\[
f(Y|X = i) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(Y - v_i)^T(Y - v_i)\right]
\]

We wish to select that value of \(X\) that maximizes \(f(Y|X = i)\). Taking natural logarithms on both sides we get:

\[
\ln f(Y|X = i) = -\frac{1}{2} \ln(2\pi) - \frac{1}{2}(Y - v_i)^T(Y - v_i)
\]

From the above equation we see that maximizing \(f(Y|X = x)\) is the same as minimizing \((Y - v_x)^T(Y - v_i)\)

\[
\hat{X} = \text{arg} \min_i [(Y - v_x)^T(Y - v_i)]
\]

The decision strategy is to pick the vector that is closest to the received vector in Euclidean distance.

b. An error occurs when the received vector \(Y\) is closer to a different vector \(v_j\) than to the transmitted vector \(v_i\).

\[
P(\text{error}) = \sum_{i=1}^{4} P(\min_{j \neq i} ||v_j - Y||^2 < ||v_i - Y||^2 | X = i) P(X = i)
\]

\[
= \frac{1}{4} \sum_{i=1}^{4} P(\min_{j \neq i} ||v_j - v_i - Z||^2 < ||Z||^2)
\]

\[
= \frac{1}{4} \sum_{i=1}^{4} P(\min_{j \neq i} ||v_j - v_i||^2 - 2(v_j - v_i)Z^T + ||Z||^2 < ||Z||^2)
\]

\[
= \frac{1}{4} \sum_{i=1}^{4} P(\min_{j \neq i} ||v_j - v_i||^2 < 2(v_j - v_i)Z^T)
\]

c. We need to choose vectors \(v_i, i \in 1, 2, 3, 4\) such that \(||v_i||^2 \leq 1\) and the Euclidean distance between any two vectors is maximal.
This implies that the four vectors must be placed on the four vertices of a square embedded in a unit circle.

To ease the decision criteria for determining $\hat{X}$ we would like four points to be equidistant orthogonal axis represented by $Y_1$ and $Y_2$.

\[
\begin{align*}
\vec{v}_1 &= (1, 1) \\
\vec{v}_2 &= (1, -1) \\
\vec{v}_3 &= (-1, -1) \\
\vec{v}_4 &= (-1, 1)
\end{align*}
\]

With this definition, we have the following decision criteria:

$\hat{X} = (1) \ 1(Y_1 > 0, Y_2 > 0) + (2) \ 1(Y_1 > 0, Y_2 < 0) + (3) \ 1(Y_1 < 0, Y_2 < 0) + (4) \ 1(Y_1 < 0, Y_2 > 0)$