Problem 1. a. 

\[ A = (3 + i)(2 - i) + 2i \]
\[ = 6 - 3i + 2i + 1 + 2i \]
\[ = 7 + i \]
\[ |A| = \sqrt{7^2 + 1^2} \]
\[ = 5\sqrt{2} \]

b. \(e^{1+z} = 2 + i\)

Let \(z = a + ib\)

\[ e^{(1+a)}e^{ib} = 2 + i \]
\[ e^{(1+a)}(\cos b + i \sin b) = 2 + i \]

From the above we obtain the following equations:

\[ e^{(1+a)} \cos b = 2 \]
\[ e^{(1+a)} \sin b = 1 \]

Dividing the two equations we get:

\[ b = \arctan \frac{1}{2} \]
\[ b = 0.4636 \]

Solving for \(a\) we get:

\[ e^{(1+a)} \cos b = 2 \]
\[ e^{(1+a)} 0.8944 = 2 \]
\[ e^{(1+a)} = 2.2361 \]
\[ 1 + a = 0.8047 \]
\[ a = -0.1953 \]

Finally \(z = -0.1953 + 0.4636i\)
Problem 2. Using the Binomial theorem:
\[
\binom{7}{0} + \binom{7}{1} + ... + \binom{7}{6} + \binom{7}{7} = (1 + 1)^7 = 128
\]

Problem 3. Since the set consists of all bijections, \( g(1) \neq g(2), g(2) \neq g(3), g(1) \neq g(3) \)

The only possible combinations are:
\[
\begin{align*}
g(1) &< g(2) < g(3) \\
g(1) &< g(3) < g(2) \\
g(2) &< g(1) < g(3) \\
g(2) &< g(3) < g(1) \\
g(3) &< g(2) < g(1) \\
g(3) &< g(1) < g(2)
\end{align*}
\]

Of the six possible cases, we are interested in the case \( g(1) < g(2) < g(3) \). Hence the probability of this case is \( \frac{1}{6} \).

Problem 4. First approach. \( \Omega \) is the universal set and \( \phi \) is the empty set. Hence by definition
\[
\begin{align*}
(A \cap A^c) &= (B \cap B^c) = \phi \\
(A \cup A^c) &= (B \cup B^c) = \Omega
\end{align*}
\]

So we have:
\[
\begin{align*}
(A \cap B) \cap (A^c \cup B^c) &= (A \cap B \cap A^c) \cup (A \cap B \cap B^c) \\
&= (B \cap \phi) \cup (A \cap \phi) \\
&= \phi
\end{align*}
\]

And:
\[
\begin{align*}
(A \cap B) \cup (A^c \cup B^c) &= (A^c \cup B^c \cup A) \cap (A^c \cup B^c \cup B) \\
&= (B \cup \Omega) \cap (A \cup \Omega) \\
&= \Omega
\end{align*}
\]

Hence the sets \((A \cap B)\) and \(A^c \cup B^c\) are mutually exclusive and their union is the universal set. Hence by definition of a complement of a set we get:
\[
(A \cap B)^c = (A^c \cup B^c)
\]

Second approach. Assume \( \omega \) is not in \( A \cap B \). That means that \( \omega \) is not both in \( A \) and in \( B \). That is, \( \omega \) is either not in \( A \) or not in \( B \). In other words, \( \omega \) is either in \( A^c \) or in \( B^c \). That is, \( \omega \) is in \( A^c \cup B^c \). We can also argue in the reverse direction. That is, assume that \( \omega \) is in \( A^c \cup B^c \). Then \( \omega \) is either in \( A^c \) or \( B^c \), ..., and therefore \( \omega \) is not in \( A \cap B \).
Problem 5. Sentences in English are finite strings of symbols taken from a finite set $A$. This set consists of the letters, punctuation marks, and the space symbol.

Define $S_1$ as the set of all sentences which consist of a single element from set $A$. This set is finite and hence countable.

Similarly define $S_m$ as the set of all sentences comprising $m$ elements from $A$. This set is also finite (it has a cardinality of $|A|^m$ elements) and hence countable.

Now construct the set $S = \cup_{n=1}^{\infty} S_n$. This set is a countable collection of countable sets and hence it is countable.

Sentences defined using elements from set $A$ may be grammatically incorrect (for example, our way of construction will allow "a,,de." to be a sentence). The set of grammatically correct sentences in English is a subset of $S$ and hence is countable.

Problem 6.

$$\int_0^1 \frac{x}{1 + x} dx = \int_0^1 1 - \frac{1}{1 + x} dx$$

$$= [x]_0^1 - [\ln(x + 1)]_0^1$$

$$= 1 - \ln2$$

Problem 7.

$$\lim_{x \to 1} = 1$$

$$\lim_{x \to 1} = 2$$

$$\sup \{g(x) | x > 1\} = 1$$

$$\inf \{g(x) | x > 0\} = -\infty$$

Problem 8. This problem can be solved using the ‘stars and bars’ method

Since the third digit takes on values from 0 to 9, the sum of the first two digits needs to be in the range 0 to 9. By using the ‘stars and bars’ method we can find the number of digits that sum to a particular value.

Let $S_n$ be the number combinations of digits that sum to $n$. Then $S_n = \binom{n+1}{1} = n + 1$

Total numbers = $\sum_{n=0}^{9} S_n = \sum_{n=0}^{9}(n + 1) = 55.$

Problem 9. Let

$$V = p + p^2 + p^3 + \cdots + p^N.$$ 

Note that $(1 - p)V = (p + p^2 + p^3 + \cdots + p^N) - (p^2 + p^3 + \cdots + p^N + p^{N+1}) = p - p^{N+1},$

so that

$$V = \frac{p - p^{N+1}}{1 - p}.$$ 

Now, let

$$S = p + 2p^2 + \cdots + Np^N.$$ 

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Then

\[ S = p + 2p^2 + \cdots + Np^N \]
\[ pS = + 1p^2 + \cdots + (N - 1)p^N + Np^{N+1} \]

Subtracting the second equation from the first we get:

\[(1 - p)S = p + p^2 + \cdots + p^N - Np^{N+1} = V - NP^{N+1} \]
\[= \frac{p - (N + 1)p^{N+1} + Np^{N+2}}{1 - p}, \]

so that

\[ S = \frac{p - (N + 1) \times p^{N+1} + N \times p^{N+2}}{(1 - p)^2}. \]

Another approach is to take the derivative with respect to \( p \) of \( 1 + p + p^2 + \cdots + p^N \) and to multiply the result by \( p \). We let you try this method.