Problem 1. (20 points) State whether the following statements are True or False. Provide reasons for your answers.

(a) (5 points) X, Y, Z are Jointly Gaussian Random variables, then X must be Gaussian.

(b) (5 points) X is Poisson(1). P(X > 10) = 0.4

(c) (5 points) X and Y are i.i.d. random variables, then L[X|Y] = E[X]

(d) (5 points) X and Y are i.i.d. N(0, 1), then X + Y and X − Y are independent.

Answer 1. (a) True. We know that X, Y, Z are Jointly Gaussian Random if and only if all linear combinations of these random variables is Gaussian.

i.e. aX + bY + cZ is Gaussian ∀a, b, c

Now if we choose a = 1, b = 0, c = 0 then we see that X has to be Gaussian.

(b) False. From Chebyshev’s inequality we know that:

\[ P(X \geq a) \leq P(|X| \geq a) \leq \frac{E[X^2]}{a^2} \]

We have \( E[X^2] = Var(X) + E[X]^2 = 1 + 1 = 2 \)

Hence,

\[ P(X \geq 10) \leq \frac{E[X^2]}{10^2} \]
\[ P(X \geq 10) \leq \frac{2}{100} \]
\[ P(X \geq 10) \leq 0.02 \]

(c) True.
\[ L[X|Y] = a(Y - E[Y]) + b \] Solving for \( a \) and \( b \) we get, \( b = E[X] \) and \( a = 0 \)

(d) True.

\( X \) and \( Y \) are individually Gaussian and independent, hence the pair \((X, Y)\) is Jointly Gaussian. Let \( V = X + Y \), \( U = X - Y \). It's easy to see that \( U \) and \( V \) are \( N(0, 2) \). Since \( U \) and \( V \) are linear combinations of Jointly Gaussian Random Variables the pair \((U, V)\) is Jointly Gaussian. Also \( \text{Cov}(UV) = E((X+Y)(X-Y)) = E[X^2] - E[Y^2] = 1 - 1 = 0 \).

Hence \( V \) and \( U \) are uncorrelated. Since they are Jointly Gaussian they are also independent.

Problem 2. (15 points) \( X \) and \( Y \) are i.i.d. \( \text{Unif}(\frac{-1}{2}, \frac{1}{2}) \) and \( Z = X^2 + Y \).

(a) (5 points) Find the conditional density \( f_{Z|X}(z|x) \).

\[ f_{Z|X}(z|x) = \begin{cases} 1 & x^2 - \frac{1}{2} \leq z \leq x^2 + \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \]

(b) (5 points) Find the MMSE estimate of \( Z \) given \( X \).

(c) (5 points) Find the expected estimation error of the MMSE estimate in part (b).

Answer 2. (a) \( f_{Z|X=x}(z|x) = \text{Unif}(x^2 - Y, x^2 + Y) \).

\[ f_{Z|X=x}(z|x) = \begin{cases} 1 & x^2 - \frac{1}{2} \leq z \leq x^2 + \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \]

b. The MMSE estimate of \( Z \) given \( X \) is \( E[Z|X] = X^2 \).

c. The Mean Square Error from (b) is: \( E[(Z - X^2)^2] = E[E[(Z - X^2)^2|X]] \).

\[ E[(Z - X^2)^2|X] = E[(Z^2 - 2ZX^2 + X^4)|X] = E[Z^2|X] - 2X^2E[Z|X] + X^4 \]
\[ = E[Z^2|X] - 2X^2E[Z|X] + X^4 \]
\[ = X^4 + \frac{1}{12} - 2X^4 + X^4 \]
\[ = \frac{1}{12} \]

Hence, \( E[(Z - X^2)^2] = E[\frac{1}{12}] = \frac{1}{12} \).

Problem 3. (15 points). \( X, Y \) and \( Z \) are jointly Gaussian random variables with mean zero and covariance matrix:

\[
\begin{bmatrix}
3 & 2 & 1 \\
2 & 3 & 2 \\
1 & 2 & 3
\end{bmatrix}
\]
Let \( U = X + Y + 2Z \) and \( V = 2X + Y + 3Z \)

(a) \(7\) points) Determine the joint density of \( U \) and \( V \)

(b) \(8\) points) Find \( E[U|V] \)

**Answer 3a.** We can write \([U \ V]^T = A[X \ Y \ Z]^T i.e.:\)

\[
\begin{bmatrix}
U \\
V
\end{bmatrix} = 
\begin{bmatrix}
1 & 1 & 2 \\
2 & 1 & 3
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
\]

Then the covariance matrix of \( U \) and \( V \) (\(C_{UV}\)) = \(ACA^T = \)

\[
\begin{bmatrix}
34 & 50 \\
50 & 74
\end{bmatrix}
\]

\(|C_{UV}| = 16, E[U] = 0, E[V] = 0 \) and

\[C_{UV}^{-1} = \begin{bmatrix}
4.625 & -3.125 \\
-3.125 & 2.125
\end{bmatrix}\]

\(f_{UV}(u, v) = \frac{1}{8\pi} e^{-\frac{1}{2}[u \ v]C_{UV}^{-1}[u \ v]^T} \)

\[= \frac{1}{8\pi} e^{-\frac{1}{2}(4.625u^2 - 6.25uv + 2.125v^2)}\]

b. We know that \(E[U|V] = a + b(V - E[V]).\) Hence \(a = 0.\) and \(b = \frac{\text{Cov}(UV)}{\text{Var}(V)} = \frac{50}{74} = 0.6757\)

**Problem 4. \(25\) points.** Let \(X\) be Uniformly distributed random variable on \([0, 1]\). Then \(X\) divides \([0, 1]\) into subintervals \([0, X]\) and \((X, 1]\). By symmetry, the length of each subinterval has mean \(\frac{1}{2}\). Now pick one of these subintervals at random in the following way. Let \(Y\) be independent of \(X\) and uniformly distributed in \([0, 1]\) and pick the subinterval \([0, X]\) or \((X, 1]\) that \(Y\) falls in. Let \(L\) be the length of the subinterval so chosen. Formally,

\[
L = \begin{cases} 
X & Y < X \\
1 - X & Y > X
\end{cases}
\]

Determine the mean of \(L\).

**Answer 4.** We have \(L = X \ 1(Y < X) + (1 - X) \ 1(Y > X)\)

Remember that \(E[L] = E[E[L|X]]\).

\(E[L|X] = X \ P(Y < X) + (1 - X) \ P(Y > X)\)

Now \(P(Y < X) = X\) and \(P(Y > X) = 1 - X\)

So \(E[L|X] = X^2 + (1 - X)^2 = 2X^2 - 2X + 1\)

\(E[L] = E[E[L|X]] = E[2X^2 - 2X + 1] = \frac{2}{3} - 1 + 1 = \frac{2}{3}\)
Problem 5. (25 points). Let X and Y be independent exponential random variables. X has a mean of \( \frac{1}{\lambda} \) and Y has a mean of \( \frac{1}{\mu} \). Z = X + Y. Determine \( E[Z^3] \).

Answer 5. \( \phi_X(\omega) = \frac{\lambda}{\lambda - j\omega} \)

\[
E[X^n] = \frac{1}{(i\pi)^n} \frac{d^n}{d\omega^n} \phi_X(\omega) |_{\omega=0} = \frac{n! \lambda}{(i\pi)(\lambda-j\omega)^{n+1}} |_{\omega=0} = \frac{n!}{\lambda^n}
\]

Similarly \( E[Y^n] = \frac{n!}{\mu^n} \)

\[
E[Z^3] = E[X^3 + 3X^2Y + 3Y^2X + Y^3]
\]

\[
\]

\[
= \frac{6}{\lambda^3} + \frac{6}{\mu \lambda^2} + \frac{6}{\mu^2 \lambda} + \frac{6}{\mu^3}
\]