Abstract

To benefit from and do well in this course, it is necessary for you to be familiar with certain concepts and skills. The objective of these notes is to list what we will assume familiar. If you have difficulties with some of these topics, you should make sure that you review them thoroughly.

1 Numbers

1.1 Real, Complex, etc

You are familiar with whole, rational, real, and complex numbers. You know how to perform operations on complex numbers and how to convert them to and from the $r \times e^{i \theta}$ notation. Recall that $|a + ib| = \sqrt{a^2 + b^2}$.

For instance, you can show that

$$\frac{3 + i}{1 - i} = 1 + 2i$$

and that

$$\sqrt{3} + i = 2e^{i \pi/6}.$$ 

1.2 Min, Max, Inf, Sup

Let $A$ be a set of real numbers. An upper bound of $A$ is a finite number $b$ such that $b \geq a$ for all $a$ in $A$. If there is an upper bound of $A$ that is in $A$, it is called the maximal element of $A$ and is designated by $\max\{A\}$. If $A$ has an upper bound, it has a lowest upper bound that is designated by $\sup\{A\}$. One defines a lower bound, the minimal element $\min\{A\}$, and the greatest lower bound $\inf\{A\}$ similarly.

For instance, let $A = \{2, 5\}$. Then 6 is an upper bound of $A$, 1 is a lower bound, $5 = \max\{A\} = \sup\{A\}$, $2 = \inf\{A\}$ and $A$ has no minimal element.

For any real number $x$ one defines $x^+ = \max\{x, 0\}$ and $x^- = (-x)^+$. Note that $|x| = x^+ + x^-$. We also use the notation $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. For instance, $(-5)^+ = 0, (-5)^- = 5, 3 \vee 6 = 6$, and $3 \wedge 6 = 3$.

2 Summations

You recall the notations

$$\sum_{n=0}^{N} x_n \text{ and } \prod_{n=0}^{N} x_n$$
and you can calculate the corresponding expressions for some specific examples of sequences \( \{x_n, n \geq 1\} \). For instance, you remember and you can prove that if \( a \neq 1 \), then
\[
\sum_{n=0}^{N} a^n = \frac{1 - a^{N+1}}{1 - a}.
\]
You also remember that
\[
\sum_{n=0}^{\infty} x_n := \lim_{N \to \infty} \sum_{n=0}^{N} x_n
\]
when the limit exists. For instance, you remember and you can prove that if \( |a| < 1 \), then
\[
\sum_{n=0}^{\infty} a^n = \frac{1}{1 - a}.
\]
By taking the derivative of the above expression with respect to \( a \), you find that, when \( |a| < 1 \),
\[
\sum_{n=0}^{\infty} na^{n-1} = \frac{1}{(1 - a)^2}.
\]
By taking the derivative one more time, we get
\[
\sum_{n=0}^{\infty} n(n-1)a^{n-2} = \frac{2}{(1 - a)^3}.
\]
It is sometimes helpful to exchange the order of summation. For instance,
\[
\sum_{n=0}^{N} \sum_{m=0}^{n} x_{m,n} = \sum_{m=0}^{N} \sum_{n=m}^{N} x_{m,n}.
\]

3 Combinatorics

3.1 Permutations

There are \( n! \) ways to order \( n \) distinct elements, where
\[
 n! = 1 \times 2 \times 3 \times \ldots \times n.
\]
By convention, \( 0! = 1 \).
For instance, there are 120 ways of seating 5 people at a table with 5 chairs.

3.2 Combinations

There are \( \binom{N}{n} \) distinct groups of \( n \) objects selected without replacement from a set of \( N \) distinct objects, where
\[
\binom{N}{n} = \frac{N!}{(N - n)!n!}.
\]
For instance, there are about \( 2.6 \times 10^6 \) distinct sets of five cards picked from a 52-card deck.
You remember that
\[
(a + b)^N = \sum_{n=0}^{N} \binom{N}{n} a^n b^{N-n}.
\]

3.3 Variations

You should be able to apply these ideas and their variations. For instance, you can count the number of strings of five letters that have exactly one E.

4 Calculus

You remember the meaning of
\[
\int_{a}^{b} f(x)dx.
\]
In particular, you know how to calculate some simple integrals. You recall that, for \( n = 0, 1, 2, \ldots \),
\[
\int_{0}^{1} x^n dx = 1/(n + 1).
\]
Also,
\[ \int_1^A \frac{1}{x} \, dx = \ln A. \]

You know the integration by parts formula and you can calculate
\[ \int_0^y x^n e^x \, dx. \]

A useful fact is that, for any complex number \( a \),
\[ (1 + \frac{a}{n})^n \to e^a \text{ as } n \to \infty. \]

You also remember that
\[ e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \]

and
\[ \ln(1 + x) \approx x \text{ whenever } |x| \ll 1. \]

5 Sets

A set is a well-defined collection of elements. That is, for every element one can determine whether it is in the set or not. Recall the notation \( x \in A \) meaning that \( x \) is an element of the set \( A \). We also say that \( x \) belongs to \( A \).

It is usual to characterize a set by a proposition that its elements satisfy. For instance one can define
\[ A = \{ x \mid 0 < x < 1 \text{ and } x \text{ is a rational number} \}. \]

You recall the definition
\[ A \cap B = \{ x \mid x \in A \text{ and } a \in B \}. \]

Similarly, you know how to define \( A \cup B, \) \( A \setminus B, \) and \( A \Delta B \). You also know the meaning of \( A \subset B \) and of the complement \( A^c \) of \( A \).

You can show that
\[ (A \cup B)^c = A^c \cap B^c \text{ and } (A \cap B)^c = A^c \cup B^c. \]

You are not confused by the notation and you would never write \([1, 2] \in [0, 3]\) because you know that \([1, 2] \subset [0, 3]\). Similarly, you would never write \(1 \in [0, 3]\) but you would write \(1 \in [0, 3]\) or \([1] \subset [0, 3]\). Along the same lines, you know that \(0 \in [0, 3]\) but you would never write \(0 \in [0, 3]\).

You could meditate on the meaning of
\[ S = \{ x \mid x \text{ is not a member of } x \}. \]

6 Countability

A set \( A \) is *countable* if it is finite or if one can enumerate its elements as \( A = \{ a_1, a_2, a_3, \ldots \} \).

A subset of a countable set is countable. If the sets \( A_n \) are countable for \( n \geq 1 \), then so is their union
\[ A = \bigcup_{n=1}^{\infty} A_n := \{ a \mid a \in A_n \text{ for some } n \geq 1 \}. \]

The *cartesian product* \( A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \} \) of countable sets is countable. The set of rational numbers is countable. The set \([0, 1] \) is not countable. (See the online notes on countability.)

7 Basic Logic

7.1 Proof by Contradiction

Let \( p \) and \( q \) be two propositions. We say “if \( p \) then \( q \)” if the proposition \( q \) is true whenever \( p \) is. For instance, if \( p \) means “it rains” and \( q \) means “the roof gets wet,” then we can postulate “if \( p \) then \( q \)”.

You know that if the statement “if \( p \) then \( q \)” is true, then so is the statement “if not \( q \) then
not $p$.” However, the statement “if not $p$ then not $q$” may not be true.

Therefore, if we know that the statement “if $p$ then $q$” is true, a method for proving “not $p$” is to prove “not $q$”.

As an example, let us prove by contradiction that the statement “$\sqrt{2}$ is irrational” is true. Let $p$ designate the statement “$\sqrt{2}$ is rational.” We know that “if $p$ then $q$” where $q$ is the statement “$\sqrt{2} = a/b$ where $a$ and $b$ are integers”. We will prove that “not $q$” is true. To do this, assume that $q$ is true, i.e., that $\sqrt{2} = a/b$. We can simplify that fraction until $a$ and $b$ are not both multiples of 2. Taking the square, we get $2 = a^2/b^2$. This implies that $a^2 = 2b^2$ is even, which implies that $a$ is even and that $b$ is not (since $a$ and $b$ are not both multiples of 2). But then $a = 2c$ and $a^2 = 4c^2 = 2b^2$, which shows that $b^2 = 2c^2$ is even, so that $b$ must also be even, which contradicts our assumption.

7.2 Proof by Induction

Assume that for $n \geq 1$, $p(n)$ designates a proposition. The induction method for proving that $p(n)$ is true for all finite $n \geq 1$ consists in showing first that $p(1)$ is true and second that if $p(n)$ is true, then so is $p(n + 1)$. The second step is called the induction step.

As an example, we show that if $a \neq 1$ then

$$a + a^2 + \cdots + a^N = \frac{(a - a^{N+1})}{(1 - a)} + a^{N+1}$$

which proves the identity for $N + 1$.

If $p(n)$ is true for all finite $n$, this does not imply that $p(\infty)$ is true, even if $p(\infty)$ is well-defined. For instance, the set $\{1, 2, \ldots, n\}$ is finite for all finite $n$, but $\{1, 2, \ldots\}$ is infinite.

8 Sample Problems

Problem 1 Express $(1 + 3i)/(2 + i)$ in the form $a + bi$ and in the form $r \times e^{i\theta}$.

Problem 2 Prove by induction that

$$\sum_{k=1}^{n} k^3 = \left( \sum_{k=1}^{n} k \right)^2.$$  

Note: We want a proof by induction, not a direct proof. You may use the fact that

$$\sum_{k=1}^{n} k = \frac{n(n + 1)}{2}.$$  

Problem 3 Give an example of a function $f(x)$ defined on $[0, 1]$ such that

$$\sup_{0 \leq x \leq 1} f(x) = 1 \quad \text{and} \quad \inf_{0 \leq x \leq 1} f(x) = 0$$

and the function $f(x)$ does not have a maximum on $[0, 1]$.

Problem 4 Calculate $\int_{0}^{1} \frac{x+1}{x+2} dx$.

Problem 5 Which of the following is/are true?

1. $0 \in (0, 1)$
2. $0 \subset (-1, 3)$
3. $(0, 1) \cup (1, 2) = (0, 2)$
4. The set of integers is uncountable.
Problem 6 Calculate \( \int_0^\infty x^2 e^{-x} \, dx \).

Problem 7 Let \( A = (1, 5), B = [0, 3), \text{ and } C = (2, 4) \). What is \( A \setminus (B \Delta C) \)?

Problem 8 Calculate
\[
\sum_{n=0}^{N} \sum_{m=n}^{N} \frac{1}{n+1}.
\]

Problem 9 Let \( A = [3, 4.7) \). What are \( \min\{A\}, \max\{A\}, \inf\{A\}, \text{ and } \sup\{A\} \)?

Problem 10 Let \( A \) be a set of numbers and define \( B = \{-a | a \in A\} \). Show that \( \inf\{A\} = -\sup\{B\} \).

Problem 11 Calculate, for \( |a| < 1 \),
\[
\sum_{n=0}^{\infty} na^n \text{ and } \sum_{n=0}^{\infty} n^2 a^n.
\]

Problem 12 How many distinct sets of five cards with three red cards can one draw from a deck of 52 cards?

Problem 13 Let \( A \) be a set of real numbers with an upper bound \( b \). Show that \( \sup\{A\} \) exists.

Problem 14 Derive the expression for \( \sum_{n=0}^{N} a^n \).

Problem 15 Let \( \{x_n, n \geq 1\} \) be real numbers such that \( x_n \leq x_{n+1} \) and \( x_n \leq a < \infty \) for \( n \geq 1 \). Prove that \( x_n \to x \) as \( n \to \infty \) for some \( x < \infty \).

Problem 16 Show that the set of finite sentences in English is countable.