This problem set essentially reviews Markov chains and Poisson Processes. Not all exercises are to be turned in. Only those with the sign ★ are due on Thursday, November 16th at the beginning of the class. Although the remaining exercises are not graded, you are encouraged to go through them.
We will discuss some of the exercises during discussion sections.
Please feel free to point out errors and notions that need to be clarified.

Exercise 7.1. ★
Consider the Markov chain \(X_n\) with the transition diagram shown in figure 7.1 where \(n > 1\) and \(p \in (0, 0.5)\)

Explain how to calculate \(P[T_n < T_0 | X_0 = 1]\) where \(T_i = \min\{n \geq 0 | X_n = i\}\). Specifically:
(a) What are the first step equations?
(b) What are the boundary conditions?
(c) Can you solve?

Solution
(a) Let \(\alpha(i) := P[T_n < T_0 | X_0 = i]\). We find
\[
\alpha(i) = p\alpha(i + 1) + (1 - p)\alpha(i - 1), 0 < i < n.
\]
(b) \(\alpha(0) = 0\) and \(\alpha(n) = 1\).
(c) Let us try \(\alpha(i) = \lambda^i\). Then
\[
\lambda^i = p\lambda \times \lambda^i + (1 - p)\lambda^{-1}\lambda^i.
\]
This is solved if \(1 = p\lambda + (1 - p)\lambda^{-1}\), i.e., if \(\lambda = 1\) or \(\lambda = (1 - p)/p\). Thus, the general solution of the equations is
\[
\alpha(i) = a + br^i \text{ where } r := \frac{1 - p}{p}.
\]
With the boundary conditions, we find \( a + b = 0 \) and \( a + b\rho^n = 1 \).
Hence,
\[
\alpha(i) = \frac{1 - \rho^i}{1 - \rho^n}, \text{ for } 0 \leq i \leq n.
\]
Finally,
\[
P[T_n < T_0|X_0 = 1] = \alpha(1) = \frac{1 - \rho}{1 - \rho^n}.
\]

Exercise 7.2. ★
A transition probability matrix \( P \) is doubly stochastic if the sum over each column equals one; that is
\[
\sum_i P_{ij} = 1, \text{ for all } j
\]
If such a chain is irreducible and aperiodic and consists of \( M + 1 \) states \( 0, 1, \ldots, M \), show that the limiting probability is given by:
\[
\pi_j = \frac{1}{M + 1}, \quad j = 0, 1, \ldots, M
\]

Solution
We know that an irreducible and aperiodic Markov chain has a limiting probability which is the unique solution of \( \pi = \pi P \). Thus we only need to check that the distribution given in this exercise is a solution.
Letting \( \pi_j = \frac{1}{M + 1}, j = 0, 1, \ldots, M \) we have
\[
(\pi P)_j = \sum_i \pi_i P_{ij}
= \frac{1}{M + 1} \sum_i P_{ij}
= \frac{1}{\pi_j}
= \frac{1}{M + 1}
\]
which gives the results.

Exercise 7.3. Let \( \pi_j \) denote the long-run proportion of time a given Markov chain is in state \( j \).
(a) Explain why \( \pi_j \) is also the proportion of transitions that are into the state \( j \) as well as being the proportion of transitions that are from state \( j \).
(b) \( \pi P_{ij} \) represents the proportion of transitions that satisfy what property?
(c) \( \sum_i \pi_i P_{ij} \) represents the proportion of transitions that satisfy what property?
(d) Using the preceding explain why
\[
\pi_j = \sum_i \pi_i P_{ij}
\]
(a) The number of transitions into \( i \), the number of transitions from \( i \), and the number of time period the process is in \( i \) are all different by at most 1, thus their long-run proportions are equal.

(b) \( \pi_i P_{ij} \) is the long-run proportion of transitions from \( i \) to \( j \).

(c) \( \sum_i \pi_i P_{ij} \) is the long-run proportion of transitions into \( j \).

(d) Since \( \pi_j \) is equal to the long-run proportion of transition into \( j \) we must have \( \pi_j \sum_i \pi_i P_{ij} \).

Exercise 7.4. ★

♣ This was a prelim question!

A fly wanders around on a graph \( G \) with vertices \( V = \{1, \ldots, 5\} \) (see figure 7.2) (a) Suppose

![Figure 7.2. A fly wanders randomly on a graph](image)

that the fly wanders as follows: if it is at node \( i \) at time \( t \), then it chooses one of the neighbors \( j \) of \( i \) uniformly at random, and then wanders to node \( j \) at time \( t + 1 \). For times \( t = 1, 2, \ldots \), let \( X_t = \{ \text{fly's position at time } t \} \). Is \( X_t \) a Markov chain? Give your arguments.

(b) Does the probability vector \( Pr(X_t = i) \) converge to a limit as the time \( t \) increases, independent of the starting position?

(c) Now suppose that a spider is sitting at node 3 and will catch the fly whenever it reaches that node. On the other hand, if the fly reaches the window (node 5), it will escape. What is the probability that the fly escapes supposing it starts at node 1?

Solution

(a) The position of the fly in the next time step only depends on the current position, so the process is a Markov chain. It is irreducible, finite (thus positive recurrent), and periodic with period 2.

(b) Starting at position 1, we know that the fly will return to that position only after an even number of time steps. Thus \( P(X_t = 1 | X_0 = 1) = 0 \) for all \( t \) odd. In the other hand if the initial position is 2, the fly will enter state 1 only after an odd number of time steps. Since the process is positive recurrent we have \( P(X_t = 1 | X_0 = 2) > 0 \) for all \( t \) odd. Thus depending on the starting point we have two different limits.

(c) We want to compute \( P(T_5 < T_3 | X_0 = 1) \) where \( T_i = \min\{t \geq 0, X_t = i\} \).

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Conditioning on the first step and using the fact that the process is a Markov chain, we have:

\[
P(T_5 < T_3|X_0 = 1) = \frac{1}{2} P(T_5 < T_3|X_0 = 2) + \frac{1}{2} P(T_5 < T_3|X_0 = 4)
\]

\[
= \frac{1}{2} \left[ \frac{1}{3} + \frac{1}{3} P(T_5 < T_3|X_0 = 1) + \frac{1}{2} P(T_5 < T_3|X_0 = 1) + \frac{1}{2} \right]
\]

\[
= \frac{1}{6} + \frac{5}{12} P(T_5 < T_3|X_0 = 1)
\]

Solving the last equation gives \( P(T_5 < T_3|X_0 = 1) = \frac{2}{7} \).

**Exercise 7.5. Gambler ruin problem.**

A gambler wins $1 at each play round, with probability \( p \) and loses $1 with probability \( 1 - p \). Different rounds are assumed to be independent. The gambler keeps playing until he either accumulate $m or loses all his money. What is the probability of eventually accumulating the target amount.

Assume that the gambler starts with \( \frac{m}{2} \).

**Solution:** Hint

The exercise is similar to exercise 1. However we provide here hints for another way of computing the desired probability.

1- Write the first step equations and the boundary equations.

2- Define \( \rho = \frac{1-p}{p} \) and \( P_i \) to be the probability of eventually accumulating the target amount given that the gambler starts with \( i \). Then write the first step equations in the form

\[
P_2 - P_1 = \rho P_1 \\
P_3 - P_2 = \rho (P_2 - P_1) = \rho^2 P_1 \\
P_4 - P_3 = \rho (P_3 - P_2) = \rho^3 P_1 \\
\vdots \\
P_m - P_{m-1} = \rho^{m-1} P_1
\]

Taking the sum of the equations we get

\[
P_m - P_1 = (\rho + \rho^2 + \cdots + \rho^{m-1}) P_1
\]

Now observing that \( P_m = 1 \) and solving for \( P_1 \), we get

\[
P_1 = \frac{1 - \rho}{1 - \rho^m} \quad \text{and} \quad P_i = \frac{1 - \rho^i}{1 - \rho^m}
\]

**Exercise 7.6. ★**

An individual possesses \( r \) umbrellas which he employs in going from home to office and vice versa. If he is at home (resp. office) at the beginning (resp. end) of a day and it is raining, then he will take an umbrella with him to the office (resp. home) if there is one to be taken.
If it is not raining, then he will not take an umbrella. Assuming that, independent of the past, it rains at the beginning (end) of a day with probability $p$.

(a) Define a Markov chain that will help you determine the proportion of time that our man gets wet.

(b) Show that there exists a limiting probability and it is given by

$$\pi_j = \begin{cases} 
\frac{1-p}{r+1-p} & \text{if } i = 0 \\
\frac{1}{r+1-p} & \text{if } i = 1, \ldots, r
\end{cases}$$

(c) What fraction of time does the man gets wet?

(d) Given $r = 3$, what value of $p$ maximizes the fraction of time the man gets wet.

Solution

(a) Let $X_n$ be the number of umbrellas at home at the beginning of day $n$. Given that number, $X_{n+1}$ depends only on whether it rains or not on that day and this is independent to everything else. Thus $X_n$ is a Markov chain. The transition probabilities are given by:

$$P(X_{n+1} = i | X_n = j) = \begin{cases} 
1-p & \text{if } j = 0 \text{ and } i = j \\
p & \text{if } j = 0 \text{ and } i = 1 \\
p(1-p) & \text{if } j = 1, \ldots, r-1 \text{ and } i \in \{j-1, j+1\} \\
1-2p(1-p) & \text{if } j = 1, \ldots, r-1 \text{ and } i = j \\
p(1-p) & \text{if } j = r \text{ and } i = r-1 \\
1-p(1-p) & \text{if } j = r \text{ and } i = r
\end{cases}$$

(b) The Markov chain is finite, irreducible, and aperiodic. It is then positive recurrent and admits a unique invariant distribution which is the solution of the balance equations

$$\pi_j = (1 - 2p(1-p))\pi_j + p(1-p)\pi_{j+1} + p(1-p)\pi_{j-1}, \quad j = 1, \ldots, r-1$$

Re-writing these equations, we get

$$\pi_j = \frac{1}{2} [\pi_{j+1} + p(1-p)], \quad j = 1, \ldots, r-1$$

For $\pi_r$ we have

$$\pi_r = p(1-p)\pi_{r-1} + (1 - p(1-p))\pi_r \Leftrightarrow \pi_r = \pi_{r-1}$$

Using the value of $\pi_{r-1}$ in the balance equations we get

$$\pi_{r-1} = \frac{1}{2} [\pi_{r-2} + \pi_r] = \frac{1}{2} [\pi_{r-2} + \pi_{r-1}]$$

$$\Leftrightarrow \pi_{r-1} = \pi_{r-2}$$

Iterating with respect to $r$ we get:

$$\pi_r = \pi_{r-1} = \cdots = \pi_1$$
Now notice that for $j = 0$,
\[ \pi_0 = (1 - p)\pi_0 + p(1 - p)\pi_1 \Leftrightarrow \pi_0 = (1 - p)\pi_1 \]

Using the fact the $\pi$ is a probability distribution we get:
\[ \sum_j \pi_j = (1 - p)\pi_1 + \pi_1 + \cdots + \pi_1 = (1 - p + r)\pi_1 = 1 \]

Thus
\[ \pi_1 = \pi_2 = \cdots = \pi_r = \frac{1}{1 - p + r} \]

And
\[ \pi_0 = (1 - p)\pi_1 = \frac{1 - p}{1 - p + r} \]

(c) Our man gets wet whenever there is no umbrella at home (w.p $\pi_0$) and it rains in the morning (w.p $p$), and whenever it rains in the afternoon (w.p $p$) and all the umbrellas are at home (i.e. it did not rain in the morning) (w.p $(1 - p)\pi_r$). Thus
\[ P_{\text{r gets wet}} = p\pi_0 + (1 - p)p\pi_r = \frac{2p(1 - p)}{1 - p + r} = Pr(p) \]

(d) For $r = 3$, we maximize this probability by taking the derivative and letting it equal to zero.
\[ \frac{\partial}{\partial p} Pr(p) = \frac{p^2 - 8p + 4}{4(1 - p)^2} = 0 \Leftrightarrow p = 4 - 2\sqrt{3} \]

Exercise 7.7. ★

Let $X$ be random variable with cdf $F_X(x)$ and pdf $f_X(x)$. We define the rate function $r_X(t)$ of $X$ as:
\[ r_X(t) = \frac{f_X(t)}{1 - F_X(t)} \]

(a) Show that $r_X(t)$ uniquely determines the distribution of $X$.

Hint: Compute $P\{X \in (t, t + \epsilon)|X > t\}$.

(b) What is $r_X(t)$ when $X \sim \text{Expo}(\lambda)$?

Solution:
(a) We will show that the distribution of $X$ is a function of the rate function. Note that
\[ P(X \in \{t, t + dt\}|X > t) = \frac{P(X \in \{t, t + dt\}, X > t)}{P(X > t)} \]
\[ = \frac{f_X(t)dt}{1 - F_X(t)} = r_X(t)dt \]
\[ = \frac{dF_X(t)}{1 - F_X(t)} = r_X(t)dt \]
Taking the integral of both sides with respect to $dt$, we have

$$-\log(1 - F_X(u)) = \int_0^s r_X(t)dt$$

Taking the exponential we have

$$F_X(s) = 1 - \exp\left\{-\int_0^s r_X(t)dt\right\}$$

(b) For an exponential random variable, it is easy to check that the rate function is constant and equal to the rate $\lambda$

**Exercise 7.8. ★**

Let $X_1$ and $X_2$ be independent exponential random variables, each having rate $\mu$. Let

$$X_{(1)} = \min\{X_1, X_2\} \quad \text{and} \quad X_{(2)} = \max\{X_1, X_2\}$$

**Compute:**

(a) $E[X_{(1)}]$

(b) $\text{Var}[X_{(1)}]$

(c) $E[X_{(2)}]$

(d) $\text{Var}[X_{(2)}]$

**Solution: Hint**

These quantities can be computed using results given in class and in the notes:

(a) $E[X_{(1)}] = \frac{1}{2\mu}$

(b) $\text{Var}[X_{(1)}] = \frac{1}{4\mu^2}$

(c) $E[X_{(2)}] = \frac{3}{2\mu}$

(d) $\text{Var}[X_{(2)}] = \frac{5}{4\mu^2}$

**Exercise 7.9. ★**

Suppose that you are the $n$'th student arriving in front of the GSI's office and somebody is already inside asking a question (thus there are $n + 1$ students). Whenever the GSI finishes his discussion with one student, the next in the line enters the office and ask another question. We assume that the time it takes to complete a question is exponential with mean $\mu$ and independent to everything else. Since it's a hot day, students will wait in the line only for an exponential random time with mean $\theta$ independently to other students (if they are not served after that time, they leave!).

We assume that the office hour is long enough for this problem (maybe infinite!).

(a) Find the probability $P_n$ that you eventually get inside the office.

(b) Find the conditional expected amount of time $W_n$ you spend waiting in the line given that you eventually get inside the office.
Solution:
The key point in this exercise is to condition on the first event after the student’s arrival and to use recursion. Note that we have \( n + 1 \) independent exponential. To simplify the notation we let \( \mu' = 1/\mu \) and \( \theta' = 1/\theta \) the rates of the exponentials.

(a) If the first event corresponds to the expiration of the exponential time of the \( n' \)th student, then the desired probability is equal to zero. Else (and this happens with probability \( \frac{(n-1)\theta' + \mu'}{n\theta' + \mu'} \)), because of the memoryless property of the exponential distribution, the system re-starts after the first departure and our student is the \( (n-1)' \)th in the new system. Thus we have:

\[
P_n = \frac{(n-1)\theta' + \mu'}{(n\theta' + \mu')} P_{n-1}
\]

Using the above result with \( n - 1 \) replacing \( n \) gives

\[
P_n = \frac{(n-1)\theta' + \mu' (n-2)\theta' + \mu'}{(n\theta' + \mu')} P_{n-2}
= \frac{(n-2)\theta' + \mu'}{n\theta' + \mu'} P_{n-2}
\]

Continuing the iterations, we will get

\[
P_n = \frac{\theta' + \mu'}{n\theta' + \mu'} P_1 = \frac{\mu'}{n\theta' + \mu'}
\]

because \( P_1 = \mu'/(\theta' + \mu') \) (the probability that the student inside the office leaves first).

(b) Note that the time until the \( n' \)th student gets inside the office is equal to the minimum of the \( n + 1 \) exponential random variable plus some additional time. But the minimum is exponential and has mean \( 1/(n\theta' + \mu') \). Since all the random variables are memoryless we have (considering the time of the first departure)

\[
W_n = \frac{1}{n\theta' + \mu'} + W_{n-1}
\]

Repeating this procedure yields to the solution

\[
W_n = \sum_{i=1}^{n} \frac{1}{i\theta' + \mu'}
\]

Exercise 7.10. Bonus★★
Let \( X(t) \) be a counting process that has the following properties:

i. \( X(t) \) has stationary increment

ii. \( X(t) \) has independent increment

iii. \( X(t) \) has isolated jumps (at most one jump can occur at any time \( t \)).
(a) Show that $X(t)$ is a Poisson process.

(b) What happens if the last property is removed?

**Solution: Hint**

(a) First verify that the Poisson process has the 3 properties (see class notes). In class we implicitly assumed isolated jumps (we drew the iid exponentials one after the other.) Since the process has independent and stationary increments, we just need to show that the number of events in any interval is a Poisson random variable that depends only on the length of the interval (then problem 7.11 can be used to show that the process is Poisson.)

A student’s answer that I found very nice!

Since the process has isolated jumps, there is at most one jump in an interval of small length $\epsilon$, and the stationary increment property implies that the probability that there is exactly one jump is proportional to the length of the interval. So we can make the approximation

\[ \Pr[X(t+\epsilon) - X(t) = 1] = 1 - Pr[X(t+\epsilon) - X(t) = 0] = \epsilon \lambda \]

Let $g(n, t) = Pr[X(t) = n]$ be the probability that there are $n$ jumps by time $t$. Then

\[
g(n, t+\epsilon) = Pr[X(t+\epsilon) = n] \\
= Pr[X(t) = n, X(t+\epsilon) - X(t) = 0] + Pr[X(t) = n-1, X(t+\epsilon) - X(t) = 1]
\]

Using the independent increment property, we can rewrite the above as

\[
g(n, t+\epsilon) = Pr[X(t) = n]Pr[X(t+\epsilon) - X(t) = 0] + Pr[X(t) = n-1]Pr[X(t+\epsilon) - X(t) = 1]
\]

which gives

\[
g(n, t+\epsilon) = (1 - \lambda \epsilon)g(n, t) + \lambda \epsilon g(n-1, t)
\]

or

\[
g(n, t+\epsilon) - g(n, t) = -\lambda g(n, t) + \lambda g(n-1, t)
\]

Letting $\epsilon \to 0$ we have

\[
\frac{\partial g(n, t)}{\partial t} = -\lambda g(n, t) + \lambda g(n-1, t)
\]

This differential equation can be solve recursively.

First for $n = 0$ we have

\[
\frac{\partial g(0, t)}{\partial t} = -\lambda g(0, t)
\]

because the probability that there is an event at $-1$ is zero. Hence solving this equation leads to

\[
g(0, t) = Ae^{-\lambda t}
\]

Since this is a probability distribution, we know that it is equal to 1 for $t = 0$, thus $A = 1$.

Assuming that $g(n-1, t)$ is the probability distribution of a Poisson random variable with rate $\lambda t$, one can easily show that

\[
g(n, t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}
\]
(b) If the isolated jumps property is removed, the argument above is no longer valid. Also we know that the isolated jumps is necessary (or needed) for a Poisson process.

**Exercise 7.11. ★**

In class (also in the lecture notes) we have defined the Poisson Process starting from iid exponential random variables.

*Here is another definition:*

The counting process \( \{ N(t), t \geq 0 \} \) is said to be a Poisson Process having rate \( \lambda \) if:

i. \( N(0) = 0 \)

ii. The process has independent increment

iii. The number of events in any interval of length \( t \) is Poisson distributed with mean \( \lambda t \)

*Show that the last definition and the one given in class are equivalent.*

**Solution: Hint**

First note that, as we have seen in class, any Poisson process has the 3 properties. Now we want to show that any process with the 3 properties is a Poisson process (in the sense of the definition given in class).

We essentially have to show that a counting process that has the 3 properties is driven by iid exponentials.

The independence is given by property (ii).

Since \( N(0) = 0 \), the probability that the first event occurs after \( t \) time units is

\[
Pr(\text{Poisson}(\lambda t) = 0) = e^{-\lambda t}
\]

which implies that the arrival time of the first event is exponential with rate \( \lambda \).

Given that the first event occurred at time \( T_1 \), the probability that the second event occurs after \( T_1 + t \) is also \( e^{-\lambda t} \) (using (ii) and (iii)) independently of the actual value of \( T_1 \).

Repeating this scheme, we see that the inter-arrival times are iid exponential random variables with rate \( \lambda \).