This problem set essentially reviews notions of Poisson Processes and continuous time Markov chains. Not all exercises are to be turned in. Only those with the sign ⋆ are due on Thursday, November 28th at the beginning of the class. Although the remaining exercises are not graded, you are encouraged to go through them. We will discuss some of the exercises during discussion sections. Please feel free to point out errors and notions that need to be clarified.

Exercise 8.1. ⋆
You toss a die repeatedly until the product of the last two outcomes is equal to 12. What is the average number of time you toss your die?

Solution
To solve this problem, we will design a Markov chain with the states given by the diagram

![State diagram for the MC in Pb8.1](image)

in figure 8.1 $S_{init} = \{0\}$, $S_1 = \{2, 3, 4, 6\}$, $S_2 = \{1, 5\}$, $S_{end} = \{12\}$, where a jump (from $S_i$ to $S_j$) means that the most recent toss gave a number in $S_i$ and the next one gives a number in $S_j$. The initial state $S_{init}$ is introduced to capture the first toss. We consider that the process stops when it arrives at state $S_{end}$.

Now the number of tosses needed for the product of the last two outcomes to be equal to 12 is equal to the time $T$ it takes our process to go from state $S_{init}$ to state $S_{end}$ and its mean is $E[T] = E[T|X_0 = S_{init}]$.

To compute $E[T]$ we use first step equations and find that

$$E[T] = 1 + \frac{2}{3}E[T|X_0 = S_1] + \frac{1}{3}E[T|X_0 = S_2]$$
We can write similar equations for $E[T|X_0 = S_1]$ and $E[T|X_0 = S_2]$ and solve a system of equations to obtain

$$E[T] = 10 + 1/2$$

Exercise 8.2. ★
You observe a realization of a Poisson arrival Process with unknown rate $\lambda \in \{\lambda_1, \lambda_2\}$ up to time $t$.
How would decide whether $\lambda = \lambda_1$ of $\lambda = \lambda_2$?
Give your solution.

Solution
We know that the number of arrivals of a Poisson Process at time $t$ has a Poisson distribution with mean $\lambda t$.
First note that $(t, N(t) = n)$ is a sufficient statistic for this detection problem (since the arrival times follows the ordered statistics of iid uniform random variables, the actual values of these arrivals is irrelevant for this problem.)
Given that $\lambda = \lambda_i$, we have

$$P[N(t) = k|i] = \frac{(\lambda_i t)^n}{n!} e^{-\lambda_i t}$$

Hence the log-likelihood ratio is given by

$$LLR(n, t) = \log \left( \frac{P[N(t) = k|1]}{P[N(t) = k|2]} \right) = -(\lambda_1 - \lambda_2)t + n \log(\lambda_1/\lambda_2)$$

And the solution will be to choose $\lambda = \lambda_1$ if $\lambda_1 > \lambda_2$ and $LLR(n, t) > 0$ (find out what to do in other cases!); which is equivalent to

$$\frac{t}{n} < \frac{\log(\lambda_1/\lambda_2)}{\lambda_1 - \lambda_2}$$

Exercise 8.3. ★
Suppose you are given a Poisson Process $(N(t))$ with rate $\lambda$ starting at time zero. A random telegraph signal $x(t) \in \{-1, 1\}$ starts at $x(0) = 1$ and switches position whenever there is an arrival of the Poisson Process.
(a) Is $X(t)$ a Markov process?
(b) Is it WSS? If not give the conditions (on $x(0)$ and $N(t)$) to make the process WSS and find $R_x(t)$ and $S_x(s)$.

Solution
The key point of this exercise is to realize that $X(t) = (-1)^{N(t)} X_0$.
(a) $X(t)$ is indeed Markov. In fact the process $X(t)$ is driven by iid exponential random variables (the inter-arrival times of the Poisson process).
(b) Note that

$$E[X(t)] = E[(-1)^{N(t)} X_0]$$
which clearly depends on the time index \( t \). To make the process WSS, we need the mean of \( X(t) \) to be a constant. The only way of having that is to choose \( X(0) \) independently to the process \( N(t) \) and to have \( E[X(0)] = 0 \); this will give

\[
E[X(t)] = E[(−1)^{N(t)}X_0] = E[(−1)^{N(t)}]E[X_0] = 0
\]

We also have to check that the auto-correlation \( R(t, \tau) = E[X(t + \tau)X(t)] \) does not depend on \( t \).

\[
R(t, \tau) = E[X(t + \tau)X(t)]
\]

\[
= E[X(0)^2(−1)^{N(t+\tau)+N(t)}]
\]

\[
= E[X(0)^2]E[(−1)^{N(t+\tau)}−N(t)(−1)^{2N(t)}]
\]

\[
= E[X(0)^2]E[(−1)^{N(t+\tau)}−N(t)]
\]

where in the last equation we use the fact that \( (−1)^{2N(t)} = 1 \).

Now we use the stationary increment of the Poisson process to conclude that

\[
R(t, \tau) = E[X(0)^2]E[(−1)^{N(\tau)}] \equiv R(\tau)
\]

which confirms that \( X(t) \) is WSS.

The actual value of \( R(\tau) \) is given by (observing that \( E[X(0)^2] = 1 \))

\[
R(\tau) = E[(−1)^{N(\tau)}] = \sum_{k=0}^{\infty} e^{-\lambda\tau} \frac{(-1)^k(\lambda\tau)^k}{k!}
\]

\[
= e^{-\lambda\tau} \sum_{k=0}^{\infty} \frac{(-\lambda\tau)^k}{k!}
\]

\[
= e^{-\lambda\tau} e^{-\lambda\tau} = e^{-2\lambda\tau}
\]

\( S_x(s) \) is the Laplace transform of \( R_x(\tau) \), so we have

\[
S_x(s) = \frac{1}{s + 2\lambda}
\]

**Exercise 8.4. ⋆**

A small bird is trying to rest between two trees that are in each side of a road. Whenever a car passes by the road, the bird wakes up and flies to the other tree (in the other side of the road).

Let \( X(t) \) be the number of cars that have disturb our bird’s sleep by time \( t \), and let \( Y(t) \in \{-1, +1\} \) be the position of the bird at time \( t \) (e.g. −1 if the bird is in the right side and +1 if it is in the left side).

(1) Car’s inter-arrival times are iid uniform in \((0, 1)\).

Is \( X(t) \) Markov, is \( Y(t) \) Markov?

(2) Car’s inter-arrival time are iid Expo(\(\lambda)\).

Is \( X(t) \) Markov, is \( Y(t) \) Markov?

Is \( X(t) \) WSS? Is \( Y(t) \) WSS? If not can you find conditions to make them WSS?
Solution
Note that the bird is just following a random telegraph process modulated buy the car arrival process.
(a) If the car’s inter-arrival times are uniform we know that the modulating process is note Markov and the random telegraph process will not be Markov.
Now that we have seen renewal processes it should not be difficult to argue that both \(X(t)\) and \(Y(t)\) are renewal processes.
(b) For exponential inter-arrival time, both \(X(t)\) and \(Y(t)\) are Markov, thanks to the memoryless property of the exponential distribution.
The condition for WWS are given in the previous problem.

Exercise 8.5. ★
A new EE student always gets lost to go from his home (in the south side of the campus) to Cory hall. To avoid getting lost again, our student decides to always take the Reverse Perimeter to come to Cory. He is told that the bus arrives at the stop with inter-arrival times that are idd \(\text{Expo}(1/15)\) minutes.
(a) The student arrives at the stop at 8am and finds that the previous bus left at 7:45am, what is the expected time of the arrival time of the next bus?
(b) Arriving at 8am, what is the expected arrival time of the previous bus?
(c) Again considering that the student arrives at 8am, what is the expected time between the arrivals of the previous and the next buses? Explain.

Solution
(a) Because of the exponential nature of the inter-arrival times, the amount of time it takes for the next bus to arrive after the student’s arrival is another exponential random time (that is independent to the amount of time since the last arrival). Thus the mean time until the next arrival is still 15min (i.e. next arrival at 8:15am).
(b) The exponential random variable is also memoryless backward. By same arguments as before we have that the mean time since the last arrival is also 15min (i.e. last arrival at 7:45am)
(c) Using (a) and (b), the expected time between the arrivals of the previous and the next buses given that the student arrives at 8am is 30min.
This is a little surprising because we know that bus inter-arrival time has mean 15min. However it should be clear to you (see course in renewal process) that this second intuition is false. The reason is that the student will very likely arrive in an large interval (i.e. long inter-arrival time). Averaging over all interval lengths, we see that the inter-arrival time will be longer.
This is called the inspection paradox and is well known in renewal process theory.

Exercise 8.6. ★
Show that if \(\{N_i(t), t \geq 0\}\) are independent Poisson processes with rate \(\lambda_i, i = 1, 2\), then \(\{N(t) = N_1(t) + N_2(t), t \geq 0\}\) is a Poisson process with rate \(\lambda_1 + \lambda_2\).
Figure 8.2. Inspection paradox for exponential distribution

Solution
The easiest way to show this is to use the definition provided in Exercise 11 in HW7. We just need to check that $N(t)$ verifies the 3 properties.

- $N(0) = 0$

- Each process $N_i(t), i = 1, 2$ has independent increment, hence the sum has independent increment.

- The sum of 2 Poisson random variables is another Poisson random variable, hence $N(t) \sim \text{Poisson}((\lambda + \lambda_2)t)$.

Thus, $N(t)$ is a Poisson process.

Exercise 8.7. ⭐
An accident occurs on each day with probability $p$ independently of the other days. Let $N(n)$ be the number of accidents that occur on the first $n$ days, and $T(r)$ denote the day of the $r$’th accident.

(a) What is the distribution of $N(n)$?

(b) What is the distribution of $T_r$?

(c) Given that $N(n) = r$, show that the unordered set of $r$ days on which the accidents occurred has the same distribution as a random selection (without replacement) of $r$ of the values $1, 2, \ldots, n$.

Solution
(a) $N(n)$ is the sum of $n$ iid Bernoulli($p$), it has a Binomial distribution with parameters $(n, p)$.

(b) To compute the distribution of $T(r)$, we notice that if $T(r) = n$, then there must be an accident on day $n$ (probability $p$) and $r - 1$ accidents in the first $n - 1$ days. The probability of having $r - 1$ accidents in the first $n - 1$ is $Pr(\text{Binom}(n - 1, p) = r - 1)$ Since accidents occur independently, we have:

$$Pr[T(r) = n] = \binom{n-1}{r-1} p^{r-1} (1-p)^{n-r} p$$
(c) The $T(1), T(3), \ldots, T(r)$ be the days where the first $r$ accidents occurred. We want to compute

$$Pr[T(1) = i_1, T(2) = i_2, \ldots, T(r) = i_r | N(n) = r] = \ldots$$  \hspace{1cm} (8.1)

$$Pr[T(1) = i_1, T(2) = i_2, \ldots, T(r) = i_r, N(n) = r]$$

$$p[N(n) = r]$$

But whenever an accident occurs, the time of the next accident is independent of the past and is only function of the relative time difference i.e

$$Pr[T(i + 1) = k | T(i) = l, T(j), j < i] = Pr[T(i + 1) = k | T(i) = l] = (1 - p)^{k-l-1}p$$

Applying this result to equation 8.2, we obtain (by successive conditioning)

$$Pr[T(1) = i_1, T(2) = i_2, \ldots, T(r) = i_r | N(n) = r] = \ldots$$

$$p[N(n) = r]$$

$$p(1 - p)^{i_1}p(1 - p)^{i_2-i_1-1} \ldots p(1 - p)^{i_r-i_r-1}(1 - p)^{n-r-i_r} \frac{p^r(1 - p)^{n-r}}{\binom{n}{r}} = \ldots$$

$$r!(n-r)!$$

$$n!$$

which is a constant that does not depend on the actual values of the $i_j$’s.

Now suppose that we have an urn with $n$ identical balls numbered 1, 2, \ldots, $n$, and let’s pick $r$ balls from the urn without replacement. The probability of any $r$-tuple $(i_1, i_2, \ldots, i_r)$ is the same and is equal to

$$Pr[i_1, i_2, \ldots, i_r] = \frac{r!(n-r)!}{n!}$$

which is equal to the probability computed above.

**Exercise 8.8. ⭐**

Consider an exponential queueing system in which there are $s$ servers available. An entering customer first waits in line and then goes to the first free server. We assume that customers arrive according to a Poisson process of rate $\lambda$ and the servers’ service times are independent and exponential with same mean $1/\mu$.

(a) What is the total departure rate?

(b) What condition(s) have to be satisfied for the queue to not blow up? Assuming these condition(s), what is the proportion of time that an arriving customer finds an empty queue?
Solution
(a) The departure depends on the number of customers being served. When \( n \) users are in the system, the departure rate is equal to the rate of the minimum of \( n \) iid exponentials of rate \( \mu \). Thus the departure rate is:

\[
\mu_t = \begin{cases} 
n\mu & \text{if } \mu \leq s \\
 s\mu & \text{if } \mu > s
\end{cases}
\]

Figure 8.3 shows the state transition diagram of the corresponding Markov chain.

(b) If the arrival rate is larger than the maximum service rate, it is obvious that the queue will blow up. In the other hand if the arrival rate is less the the maximum service rate, we expect the queue to increase and shrink but never blow up, thus the condition for stability is \( \lambda < s\mu \).

Given this condition, we know the fraction of time that an arriving customer finds an empty queue is equal \( \pi_0 \) where \( \pi \) is the stationary distribution.

Using the cut-set theorem we have

\[
\lambda \pi_{i-1} = i \mu \pi_i, \quad \forall i = 1, 2, \ldots, s
\]

This gives

\[
\pi_i = \lambda \frac{\lambda}{i \mu} \pi_{i-1} = \cdots = \frac{1}{i!} \left( \frac{\lambda}{\mu} \right)^i \pi_0
\]

Taking the sum of all the \( \pi_i \), we have:

\[
1 = \pi_0 \sum_{j=0}^{s} \frac{1}{j!} \left( \frac{\lambda}{\mu} \right)^j
\]

This gives the following:

\[
\pi_0 = \frac{1}{\sum_{j=0}^{s} \frac{1}{j!} \left( \frac{\lambda}{\mu} \right)^j}
\]

Exercise 8.9. ★

Problem 18.1 of the course notes.
Solution
First notice that the CTMC is a process that, whenever in state $i$, spends there an exponential amount of time with rate $q(i) = -Q(i, i)$ and jump to state $j$ with probability $P_{ij}$. From the transition matrix $Q$ we can compute $q(i)$ and $P_{ij}$:

$$P_{ij} = \frac{Q(i, j)}{q(i)}.$$ 

Thus to simulate the Markov chain, we will proceed as follows

- Choose an initial state according to some initial distribution $\pi = [\pi_1, \pi_2, \ldots]$, where $\pi_i$ is the probability that the process starts at state $i$.
- Whenever the process enters state $i$, draw an independent exponential random time $\tau_i$ with rate $q(i)$ and let the process remain in state $i$ for the next $\tau_i$ time units.
- After $\tau_i$ time units, go back to the first step with $\pi$ replaced by $P_i = [P_{i1}, P_{i2}, \ldots, P_{ij}, \ldots]$.

Exercise 8.10. ⭐
Problem 18.2 of the course notes.

Solution
The rate matrix of the chain is

$$Q = \begin{pmatrix}
-(\lambda + \mu) & \lambda & \mu \\
0 & -\mu & \mu \\
\lambda & 0 & -\lambda
\end{pmatrix}$$

The Markov chain is finite and irreducible: it is positive recurrent and has a unique invariant distribution. This distribution satisfies the balanced equations.

$$-(\lambda + \mu)\pi(0) + \lambda\pi(2) = 0$$
$$\lambda\pi(0) - \mu\pi(1) = 0$$
$$\mu\pi(0) + \lambda\pi(1) - \lambda\pi(2) = 0$$

and

$$\pi(0) + \pi(1) + \pi(2) = 1$$

Solving these equations gives

$$\pi(0) = \frac{\lambda\mu}{(\lambda + \mu)^2} \quad \pi(1) = \frac{\lambda}{(\lambda + \mu)^2} \quad \pi(2) = \frac{\mu}{\lambda + \mu}$$

Exercise 8.11. ⭐
Problem 19.3 of the course notes.
Solution

Let us show the direct implication first.

Assume that the CTMC is time reversible, and let $i_0, i_1, \ldots, i_n$ be a sequence of states, we then have

\[
\pi(i_0)q(i_0, i_1) = \pi(i_1)q(i_1, i_0)
\]
\[
\pi(i_1)q(i_1, i_2) = \pi(i_2)q(i_2, i_1)
\]
\[
\vdots
\]
\[
\pi(i_{n-1})q(i_{n-1}, i_n) = \pi(i_n)q(i_n, i_{n-1})
\]
\[
\pi(i_n)q(i_n, i_0) = \pi(i_0)q(i_0, i_n)
\]

Taking the product of both sides of the equalities and canceling out the $\pi(\cdots)$ terms we obtain

\[
q(i_0, i_1)q(i_1, i_2) \cdots q(i_{n-1}, i_n) = q(i_0, i_n)q(i_n, i_{n-1}) \cdots q(i_1, i_0)
\]

To prove the other direction of the implication, we assume that $\pi(\cdots)$ has the form given in the hint and we consider the path

$\pi r, i_0, i_1, \ldots, i_m, i, j, j_n, j_{n-1}, \ldots, j_0, r$

that goes through states $i$ and $j$.

We have

\[
\pi(j) = \frac{\alpha}{q(j, j_n)q(j_n, j_{n-1}) \cdots q(j_0, r)} q(r, j_0)q(j_0, j_1) \cdots q(j_n, j)
\]
\[
\pi(i) = \frac{\alpha}{q(i, i_m)q(i_m, i_{m-1}) \cdots q(i_0, r)} q(r, i_0)q(i_0, i_1) \cdots q(i_m, i)
\]

Let us compute the ration $\pi(i)/\pi(j)$.

\[
\frac{\pi(i)}{\pi(j)} = \frac{q(r, j_0)q(j_0, j_1) \cdots q(j_n, j)}{q(j, j_n)q(j_n, j_{n-1}) \cdots q(j_0, r)} \frac{q(i, i_m)q(i_m, i_{m-1}) \cdots q(i_0, r)}{q(r, i_0)q(i_0, i_1) \cdots q(i_m, i)}
\]

By assumption we have

\[
\frac{q(r, j_0)q(j_0, j_1) \cdots q(j_n, j)}{q(j, j_n)q(j_n, j_{n-1}) \cdots q(j_0, r)} \frac{q(i, i_m)q(i_m, i_{m-1}) \cdots q(i_0, r)}{q(r, i_0)q(i_0, i_1) \cdots q(i_m, i)} = 1
\]

Thus we obtain

\[
\frac{\pi(i)}{\pi(j)} = \frac{q(j, i)}{q(i, j)}
\]

which indicates that the process is time reversible.
Exercise 8.12. ★
Problem 19.5 of the course notes.

Solution
To use Kelly’s Lemma we will first compute \( Q' \) and guess \( \pi \) (it is given!), then we will verify that \( Q, Q' \), and \( \pi \) verify equation 20.1 of the notes.

We have
\[
R = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
p & (1 - p) & 0
\end{pmatrix}
\]
\[
R' = \begin{pmatrix}
0 & 0 & 1 \\
p & 0 & (1 - p) \\
0 & 1 & 0
\end{pmatrix}
\]

It is easy to check that
\[
\lambda_1 = p \lambda_2, \quad \lambda_2 = \lambda_3
\]

\[
Q = \begin{pmatrix}
-\mu_1 & \mu_1 & 0 \\
0 & -\mu_2 & \mu_2 \\
p \mu_3 & (1 - p) \mu_3 & -\mu_3
\end{pmatrix}
\]
\[
Q' = \begin{pmatrix}
-\mu_1 & 0 & \mu_1 \\
p \mu_2 & -\mu_2 & (1 - p) \mu_2 \\
0 & \mu_3 & -\mu_3
\end{pmatrix}
\]

Now let’s check that \((Q, Q', \pi)\) verifies equation 20.1.

Exercise 8.13. ★
Problem 19.6 of the course notes.

Solution
(a) \( X_t \) is not a Markov chain. In fact, since the arrival rate depends on time, the observation up to time \( t \) is relevant in guessing the queue length after \( t \). For example consider the extreme case where \( \lambda_0 \sim 0, \lambda_1 >> 0 \) and \( \alpha \approx \beta \). If we observe an empty queue for a long time, it is very likely that \( Y = 0 \) and we expect the queue to be empty for an exponential additional
time (the time for $Y$ to jump to 1). (In the other hand if we observe a non-empty queue $Y$ is likely to be 1 and we expect arrivals for an additional exponential time.)

(b) To show that $Z_t$ is Markov we will show that

$$Pr[(X_{t+\epsilon}, Y_{t+\epsilon})|(X_t, Y_t), (X_s, Y_s), s < t] = Pr[(X_{t+\epsilon}, Y_{t+\epsilon})|(X_t, Y_t)]$$

Notice that:
1- $Y_t$ is a Markov chain, thus
$$Y_{t+\epsilon} \perp \perp Y_s, s < t | Y_t$$
also, $Y_{t+\epsilon}$ does not depend on $X_s, s \leq t$.
2- Given $Y_t$ (⇒ the arrival rate) and $x_t$, the system will look like an MM1 queue for the next $\epsilon$ time unit, thus
$$X_{t+\epsilon} \perp \perp (X_s, Y_s), s < t | (X_t, Y_t)$$
Combining 1- and 2-, we can conclude that
$$(X_{t+\epsilon}, Y_{t+\epsilon}) \perp \perp (X_s, Y_s), s < t | (X_t, Y_t)$$

(c) Note that the process $(Y_t), t \geq 0$ is positive recurrent, so we only need to check the drift of $X_t$.

$$E[X(t+\epsilon) - X(t)|X(t) = i, Y(t) = j, j = 0, 1] = E[X(t + \epsilon)|X(t) = i, Y(t) = j, j = 0, 1] - i$$
$$= (i + 1)\frac{\lambda_j}{\lambda_j + \mu} + (i - 1)\frac{\mu}{\lambda_j + \mu} - i$$
$$= i(\frac{\lambda_j}{\lambda_j + \mu} + \frac{\mu}{\lambda_j + \mu} - 1) + \frac{\lambda_j - \mu}{\lambda_j + \mu}$$
$$= \frac{\lambda_j - \mu}{\lambda_j + \mu}, j = 0, 1$$

For the drift to be always negative, we must have
$$\mu > \max(\lambda_0, \lambda_1)$$