This problem set essentially reviews Convergence and Renewal processes. Not all exercises are to be turned in. Only those with the sign ⭐ are due on Thursday, December 7th at the beginning of the class. Although the remaining exercises are not graded, you are encouraged to go through them.

We will discuss some of the exercises during discussion sections. Please feel free to point out errors and notions that need to be clarified.

Exercise 9.1. ⭐
Assume that $X_n$ converges in probability to $X$ and $f(\cdot)$ is a continuous bounded function. Show that $f(X_n)$ converges in probability to $f(X)$

Solution:
We want to show that
\[
\forall \epsilon > 0 \quad Pr\{\omega : |f(X_n(\omega)) - f(X(\omega))| > \epsilon\} \to 0
\]

$X_n$ converges in probability to $X$ implies that
\[
\forall \epsilon > 0 \quad Pr\{\omega : |X_n(\omega) - X(\omega)| > \epsilon\} \to 0
\]

Since $f$ is continuous, we also have that for all $\omega$ and $\epsilon', \delta' > 0$ there is an $n(\omega) > 0$ such that
\[
|X_n(\omega) - X(\omega)| < \delta' \Rightarrow |f(X_n(\omega)) - f(X(\omega))| < \epsilon'
\]
or
\[
|f(X_n(\omega)) - f(X(\omega))| \geq \epsilon' \Rightarrow |X_n(\omega) - X(\omega)| \geq \delta'
\]

Thus, letting $N = \max\{n(\omega)\}$, we have for $n > N$
\[
\{\omega : |f(X_n(\omega)) - f(X(\omega))| \geq \epsilon'\} \subset \{\omega : |X_n(\omega) - X(\omega)| \geq \delta'\}
\]
and
\[
P\{\omega : |f(X_n(\omega)) - f(X(\omega))| \geq \epsilon'\} \leq P\{\omega : |X_n(\omega) - X(\omega)| \geq \delta'\} \to 0
\]

which ends the proof.

Exercise 9.2. Bonus ⭐⭐
Assume that $X_n$ converges in distribution to some random variable $X$. Show that we can find $(Y_n, Y)$, where $Y_n$ is a sequence of random variables such that $Y_n$ has same distribution as $X_n$, $Y$ has the same distribution as $X$, and $Y_n$ converges almost surely (a.s) to $Y$. 

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Solution:
Let \( \Omega = (0, 1) \) be our event space and for \( \omega \) uniformly picked in \( \Omega \) let \( Y_n(\omega) = \sup \{ x : F_{X_n}(x) < \omega \} \). \( Y_n \) has distribution \( F_{X_n} \). We want to show that \( Y_n(\omega) \to Y(\omega) \) for all but a countable number of \( \omega \)'s, where \( Y \approx_d X \). To show that, let us define for all \( \omega \),
\[
\begin{align*}
a_{\omega} &= \sup \{ x : F_X(x) < \omega \}, \\
b_{\omega} &= \inf \{ x : F_X(x) > \omega \}, \\
\Omega_o &= \{ \omega : (a_{\omega}, b_{\omega}) = \emptyset \}
\end{align*}
\]
where \( (a_{\omega}, b_{\omega}) \) is an open interval with the given end point. In words, \( \Omega_o \) contains all \( \omega \in \Omega \) such that \( \liminf F_X^{-1}(\omega) = \limsup F_X^{-1}(\omega) \). Now let \( Y(\omega) = F_X^{-1}(\omega), \forall \omega \in \Omega_o \).

\( \Omega - \Omega_o \) is countable since the intervals \( (a_{\omega}, b_{\omega}) \) are disjoint (because of the use of \( \sup \) and \( \inf \) and each nonempty interval contains a different rational number; recall that the set of rational numbers is countable).

Now for any \( \omega \in \Omega_o \)
\[
\sup F_{X_n}^{-1}(\omega) \leq F^{-1}(\omega) \leq \inf F_{X_n}^{-1}(\omega)
\]
Taking the limit \( n \to \infty \) we have \( Y_n(\omega) = F_{X_n}^{-1}(\omega) \to Y(\omega), \forall \omega \in \Omega_o \).

Exercise 9.3. ★
In the notes we have shown that if \( X_n \) converges in probability to \( X \), then it converges in distribution to \( X \).
Show that, conversely, if \( X_n \) converges in distribution to a constant \( C \), then it converges in probability to \( C \).

Solution:
Since \( X_n \) converges in distribution to \( C \), we have
\[
F_{X_n}(x) \to \delta(x > C), \forall x
\]
Now let us compute
\[
Pr[|X_n - C| > \epsilon] = 1 - Pr[|X_n - C| \leq \epsilon] = 1 - Pr[C - \epsilon < X_n < C + \epsilon] = 1 - (F_{X_n}(C + \epsilon) - F_{X_n}(C - \epsilon)) \xrightarrow{n \to \infty} 1 - (1 - 0)
\]
Thus \( X_n \) converges in probability to \( C \).

Exercise 9.4. ★
Problem 21.2 of the course notes.

Solution:
Note that since \( \epsilon_n \downarrow 0 \), for all \( \epsilon > 0 \) there exists \( n_0(k,m) \geq 1 \) such that \( \epsilon_n < \epsilon \) and for all \( k, m \geq n \geq 1 \)
\[
Pr[|X_k - X_m| > \epsilon] \leq Pr[|X_k - X_m| > \epsilon_n] \leq 2^{-n}
\]
Let $Z_n(k, m) = |X_k - X_m|$ for all $k, m \geq n \geq 1$. Then

$$
\sum_n Pr[Z_n(k, m) > \epsilon] = \sum_{n \leq n_0(k, m)} Pr[Z_n(k, m) > \epsilon] + \sum_{n > n_0(k, m)} Pr[Z_n(k, m) > \epsilon]
\leq \sum_{n \leq n_0(k, m)} Pr[Z_n(k, m) > \epsilon] + \sum_{n > n_0(k, m)} 2^{-n}
< \infty
$$

Thus Borel-Cantelli Lemma implies that $Pr[Z_n(k, m) > \epsilon, i.o] = 0$. This tells that for all $\epsilon > 0$, and for all $\omega \in \Omega$, there exists $n(\epsilon, \omega) = \max\{n_0(k, m)\} > 0$ such that

$$
|X_k(w) - X_m(w)| < \epsilon, \forall k, m \geq n(\epsilon, \omega)
$$

So

$$
\sup_{m,k} |X_k(w) - X_m(w)| < \epsilon, \forall k, m \geq n(\epsilon, \omega)
$$

Hence $X_n$ is Cauchy a.s.

Since the set of real number is complete, $X_n$ converges almost surely to a limit in $R$.

**Exercise 9.5. ★**

Problem 21.9 of the course notes.

**Solution:**

For any given state $i$, let $N_i(t)$ be the process that count the number of arrivals when the CTMC $X(t)$ is in state $i$. $N_i(t)$ is a Poisson process with rate $\lambda_i$.

The proportion of time that $X(t)$ is in state $i$ is given by

$$
\pi_i = \lim_{t \to \infty} \frac{1}{t} \int_0^\infty 1_{[X_s = i]} ds
$$

So

$$
\frac{N_i(t)}{t} \to \lambda_i \pi_i
$$

We also have that

$$
N(t) = \sum_i N_i(t)
$$

Hence

$$
\lim_{t \to \infty} \frac{N(t)}{t} = \lim_{t \to \infty} \frac{1}{t} \sum_i N_i(t)
\leq \sum_i \lim_{t \to \infty} \frac{N_i(t)}{t}
= \sum_i \lambda_i \pi_i
$$

(9.1)
where to get equation 9.1 we use the fact that $X(t)$ is positive recurrent and does not explode, thus $\frac{N(t)}{t}$ must be finite (thus it is bounded and we can use the dominated convergence theorem to justify swapping the lim and the sum).

Exercise 9.6. ★
Recall that in the study of renewal process we defined the inter-arrival times $T = T_{i+1} - T_i, i \geq 1$ to be iid with distribution $F(t)$.
Let
$$f(t) = \frac{1}{(1+t)^2}$$
to be the corresponding pdf.
Find $E[\tau]$ where $\tau$ is the time until the next jump for the stationary process, and $\lambda$ the rate of jumps.
This exercise misses the point it was supposed to make. Answer the next question:
Find a distribution for which $0 < \lambda < \infty$ and $E[\tau] = \infty$.

Solution: Hint
A simple example is
$$f(t) = \frac{c}{(1+t)^3}$$
Find the constant c and see next exercise for a more interesting discussion.

Exercise 9.7. ★
In the derivation of $E[\tau]$ in class we wrote
$$E[\tau] = \int_0^\infty \lambda t(1 - F(t))dt$$
$$= \lambda \int_0^\infty (1 - F(t))dt^2$$
$$= \frac{\lambda}{2} [t^2(1 - F(t))]_0^\infty + \frac{\lambda}{2} \int_0^\infty t^2 f(t) dt$$
(9.2)
We claimed that the first term in the RHS of equation 9.2 vanishes because the mean of $T = T_2 - T_1$ (inter-arrival time) should be finite.
This argument is not quite correct; show the correct argument that is:
$t^2(1 - F(t)) \to 0$ as $t \to \infty$ if and only if $E[T^2] < \infty$.

Solution:
Thanks to all the student for being cautious/pessimistic in this exercise.
$t^2(1 - F(t)) \to 0$ as $t \to \infty$ if $E[T^2] < \infty$.
One counterexample is:
$$F(t) = 0, t < 0, \quad \text{and} \quad F(t) = 1 - \frac{c}{(t+1)^2 \ln(t+1)}, t \geq 0$$
It is easy to verify that $t^2(1 - F(t)) \to 0$ as $t \to \infty$ if $E[T^2] = \infty$. If however $E[T^2] < \infty$ we have

$$t^2(1 - F(t)) = t^2 \int_t^\infty f(s)ds = \int_t^\infty t^2 f(s)ds \leq \int_t^\infty s^2 f(s)ds \text{ bce } t < s$$

But since $\int_0^\infty s^2 f(s)ds < \infty$

$$\lim_{t \to \infty} \int_t^\infty s^2 f(s)ds = 0$$

Exercise 9.8. **Bonus** ★★★
Considering again the renewal process setting given in class, show that if the inter-arrival times are iid uniform in $[0, 1]$, then $\epsilon$-coupling occurs in finite time.

**Solution:**
To show this we will start two processes, one ($N_s(t)$) with the stationary distribution as initial distribution, and the other ($N_x(t)$) with some other initial distribution, then we will show that at some particular times (e.g. just after jumps of $N_x(t)$), there is a positive probability that the 2 processes jump within and interval of length $\epsilon$.

Let the jump times of the process $N_x(t)$ denoted $T_1, T_2, \ldots$ and consider some particular time $T_i$. Let $a \in (0, 1)$ be the time spent by process $N_s(t)$ since the most recent jump and let’s compute the probability that there will be a jump in the next $\epsilon$ second ($T_i, T_i + \epsilon$).

Given that the process $N_s(t)$ did not jump in the first $a$ second, the time arrival of the next jump is uniformly distributed in $(a, 1)$. Thus the probability that this jump happens in the next $\epsilon$ second is

$$Pr[\tau \in (T_i, T_i + \epsilon)] = Pr[\tau \in (a, 1)|\tau > a] = \frac{\epsilon}{1 - a} > \epsilon$$

Hence the probability that the two processes do not jump within $\epsilon$ second is less than $1 - \epsilon$. Since we have a renewal process and there is an infinite number of jumps, the probability that the 2 processes do not couple in finite time is

$$Pr[\text{not couple}] = (1 - \epsilon)^\infty = 0$$

Thus with probability 1, there is $\epsilon$-coupling in finite time.

Exercise 9.9. ★
In class we have shown that for a positive recurrent continuous-time Markov chain with rate matrix $Q$, and invariant distribution $\pi$, we have

$$\lim_{T \to T} \frac{1}{T} \int_0^T 1_{[x_t=i]}dt \overset{a.s.}{\to} \sum_j 1_{[j=i]}\pi_j = \pi_i$$

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Let
\[ f : \mathcal{X} \to R \]
be a bounded function from the state space \( \mathcal{X} \) to the real line \( R \).
Show that
\[ \frac{1}{T} \int_0^T f(x_t) dt \xrightarrow{as} \sum_j f(j) \pi_j \]

**Solution:**
Note that we have
\[
\frac{1}{T} \int_0^T f(x_t) dt = \frac{1}{T} \int_0^T \sum_j f(j) 1_{x_t = j} dt
\]
\[ = \sum_j f(j) \frac{1}{T} \int_0^T 1_{x_t = j} dt \quad (9.3) \]

We know that
\[
\frac{1}{T} \int_0^T 1_{x_t = i} dt \xrightarrow{as} \pi_i
\]

Taking the limit as \( T \to \infty \) in equation 9.3, we have for any finite state Markov chain with number of states \( N \)
\[
\sum_{j=1}^N f(j) \frac{1}{T} \int_0^T 1_{x_t = j} dt \xrightarrow{as} \sum_{j=1}^N f(j) \pi_j
\]
because \( f(\cdot) \) is a bounded function, so we can safely change the order of the limit and the summation using the Dominated Convergence Theorem.
For a general positive recurrent Markov chain, we will use some kind of *truncation argument*.
The intuition is that for \( T \) large enough, there is a finite number of visited states, and all states \( j \) that have not been visited must have very small \( \pi_j \). Let \( N_T \) be the set of states visited by time \( T \) we have:
\[
\sum_{j=1}^\infty f(j) \frac{1}{T} \int_0^T 1_{x_t = j} dt = \sum_{j \in N_T} f(j) \frac{1}{T} \int_0^T 1_{x_t = j} dt
\]
and
\[
\sum_{j=1}^\infty f(j) \pi_j = \sum_{j \in N_T} f(j) \pi_j + \sum_{j \notin N_T} f(j) \pi_j
\]
We just need to show that the second term in the RHS goes to zero as \( T \to \infty \). Now notice that:
1- $N_T$ is an increasing set
2- The states in $N_T$ are such that

$$\sum_{j \in N_T} f(j) \pi_j \leq M \sum_{j \in N_T} \pi_j \xrightarrow{T \to \infty} 0$$

where we have used the facts that $f(\cdot)$ is bounded and that the states in $N_T$ have smaller and smaller $\pi_j$'s.
Combining this with the previous remark (finite state), we get the result.

Exercise 9.10. ★
In Prof. X’s group, John Lazy, a very daydreaming network manager has set up a printer without queue. Any request that finds the printer busy (i.e. already printing) is just lost. Assume that requests arrive at the printer according to a Poisson process with rate $\lambda$, and the amount of time needed to print a request is a random variable having distribution $G$ with mean $\mu_G$ and independent for each request.
(a) What is the rate at which requests are accepted (i.e. requests get printed)?
(b) What is the proportion of satisfied requests?
Compute it for $\lambda = 2$ requests per second and $\mu_G = 2$ seconds.

Solution:
(a) Because of the memoryless property of the Poisson process, the mean time between entering requests is

$$\mu = \frac{1}{\lambda} + \mu_G$$

(mean time it takes for a request to arrive plus mean service time).
Hence the rate at which requests are accepted is

$$\frac{1}{\mu} = \frac{1}{1 + \lambda \mu_G}$$

(b) Requests arrive at rate $\lambda$ and are accepted with rate $1/\mu$. So the fraction of accepted request is given by

$$f = \frac{1/\mu}{\lambda} = \frac{\lambda \frac{1}{\mu_G}}{1 + \lambda \mu_G} = \frac{1}{1 + \lambda \mu_G}$$

For $\lambda = \mu_G = 2$ we have $f = 1/5$ meaning that 1 out of 5 requests is accepted.

GOOD LUCK!