EE226a - Summary of Lecture 16
Markov Chains

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I. SUMMARY

The book [1] is very clear, so my notes will be very succinct. You can also check [2] for examples. Here are the key ideas of this lecture.

- Definition of Markov chain; irreducible; period, transient.
- Examples; Random Walk.

II. DEFINITIONS

We define the following concepts.

- Markov chain $X = \{X_n, n \geq 0\}$ on $\mathcal{X}$ finite or countable; transition probability matrix $P$. The possible values of $X$ are called states.
- We define irreducible.
- We define the period of a state.

We state

**Theorem 1:** If a Markov chain is irreducible, then all the states have the same period. We define aperiodic.

III. EXAMPLES

We look at examples of Markov chains and examine whether they are irreducible and aperiodic.

IV. TRANSIENT

We define transient state:

**Definition 1:** State $i$ is transient if

$P\left[ \sum_{n=0}^{\infty} 1\{X_n = i\} < \infty | X_0 = i \right] = 1.$

We state the following result that we leave you as an exercise to prove:

**Fact 1:** An irreducible finite Markov chain cannot have transient states.

We look at one example:

**Fact 2:** Assume $P[X_{n+1} = i + 1 | X_n = i] = p$ and $P[X_{n+1} = i - 1 | X_n = i] = q = 1 - p$ for $i \in \mathcal{X} = \mathbb{Z}$. Then all the states are transient if $p \neq 1/2$.

**Proof:**

Note that $X_{n+1} = X_n + V_n$ where $\{V_0, V_n, n \geq 0\}$ are independent and $P(V_n = 1) = p = 1 - P(V_n = -1)$. Hence,

$$X_n = \frac{V_0 + V_1 + \ldots + V_{n-1}}{n} \rightarrow E(V_1) = p - q \neq 0.$$ 

This shows that $X_n \rightarrow +\infty$ if $p > 1/2$ and $X_n \rightarrow -\infty$ if $p < 1/2$. Now, if $X_n(\omega) \rightarrow \infty$, this implies that $X_n > 10$ for all $n \geq n(\omega)$ where $n(\omega) < \infty$. Consequently,

$$\sum_{n \geq 0} 1\{X_n = 0\} < n(\omega) < \infty.$$ 

Since this happens with probability one, we see that

$$P\left[ \sum_{n \geq 0} 1\{X_n = 0\} < n(\omega) < \infty | X_0 = 0 \right],$$

which shows, by definition, that 0 is transient. By symmetry, all states are transient. The same argument applies for $p < 1/2$.

REFERENCES