I. SUMMARY

Here are the key ideas of this lecture.

- Function of Markov chain.
- First Passage Time; Kolmogorov Equations
- Positive Recurrent: State
- Generally, a function of a Markov chain is not a Markov chain.
- Infinitely Often:
- Kolmogorov Equations
- Null Recurrent: State
- Reflected Random Walk.
- Transient: State

II. FUNCTION OF MARKOV CHAIN

Fact 1: Generally, a function of a Markov chain is not a Markov chain.

We look at a few examples.

Example 1: Consider the Markov chain $X$ with $X_{n+1} = (X_n + 1) \mod(3)$ where $X_0$ is uniformly distributed in $\{0, 1, 2\}$. Let $f(x) = 0$ for $x = 0, 1$ and $f(2) = 1$. Then $Y_n = f(X_n)$ is not a Markov chain. Indeed,

$$P[Y_2 = 0|Y_1 = 0, Y_0 = 0] = 0 \neq P[Y_2 = 0|Y_1 = 0] = \frac{1}{2}.$$ 

Intuitively, $f(X_n)$ contains less information than $X_n$ and it may happen that $\{f(X_n), m \leq n\}$ has information about $X_n$ that $f(X_n)$ does not contain, as the example shows.

Example 2: As a trivial example where $f(X_n)$ is a Markov chain even though $f(\cdot)$ is not one-to-one, take $f$ to be constant.

III. DEFINITIONS

Definition 1: We define

- First Passage Times: $\tau_i := \min\{n > 0|X_n = i\}; T_i := \min\{n \geq 0|X_n = i\}$. We extend these definitions to $T_S$ and $\tau_S$ for $S \subseteq X$.
- Transient: State $i$ is transient if $P[\tau_i < \infty|X_0 = i] < 1$.
- Recurrent: State $i$ is recurrent if $P[\tau_i < \infty|X_0 = i] = 1$.
- Positive Recurrent: State $i$ is PR if $E[\tau_i|X_0 = i] < \infty$.
- Null Recurrent: State $i$ is NR if it is recurrent and $E[\tau_i|X_0 = i] = \infty$.
- Infinitely Often: $\{X_n = i, \text{i.o.}\} = \{\omega|X_n(\omega) = i\}$ for infinitely many $n'$s $= \{\omega|\sum_{n=0}^{\infty}1\{X_n(\omega) = i\} = \infty\}$.

IV. KOLMOGOROV EQUATIONS

First passage times satisfy simple first step equations that we can use for their analysis.

Fact 2: Kolmogorov Equations

(a) One has, for any $A, B \subseteq X$ and $i \in X$,

$$\alpha(i) := P[T_A < T_B|X_0 = i] = \begin{cases} \sum_j P(i, j)\alpha(j), & \text{if } i \notin A \cup B; \\
1, & \text{if } i \in A \setminus B; \\
0, & \text{if } i \in B. \end{cases}$$

(b) One has, for any $A \subseteq X$ and $i \in X$,

$$\beta(i) := E[T_A|X_0 = i] = \begin{cases} 1 + \sum_j P(i, j)\beta(j), & \text{if } i \notin A; \\
0, & \text{if } i \in A. \end{cases}$$

(c) Assume that $X = Z$ or $X = Z_+$ and that $T_i \uparrow \infty$ as $i \to \infty$. Then, if $i \notin A$,

$$P[T_A < T_i|X_0 = j] \uparrow P[T_A < \infty|X_0 = j] \quad \text{as } i \to \infty$$

and

$$E[\min\{T_A, T_i\}|X_0 = j] \uparrow E[T_A|X_0 = j] \quad \text{as } i \to \infty.$$ 

Proof:

Everything is obvious, except for (c) which relies on part (a) of the following result.
Theorem 1: Lebesgue Convergence Theorem
(a) Assume that \(0 \leq X_n \uparrow X\) as \(n \to \infty\). Then \(E(X_n) \uparrow E(X)\) as \(n \to \infty\).
(b) Assume that \(X_n \to X\) as \(n \to \infty\) and \(|X_n| \leq Y\) with \(E(Y) < \infty\). Then \(E(X_n) \to E(X)\) as \(n \to \infty\).

Remark. You might be tempted to think that \(X_n \to X\) as \(n \to \infty\) implies \(E(X_n) \to E(X)\) as \(n \to \infty\). However, this is not true as the following example illustrates.

Example 3: Let \(\Omega = (0, 1]\) and assume that \(\omega\) is picked uniformly in \(\Omega\). For \(n \geq 1\), let \(X_n(\omega) = n1\{\omega \leq 1/n\}\). Then \(P(X_n = n) = 1/n = 1 - P(X_n = 0)\), so that \(E(X_n) = n \times (1/n) = 1\) for \(n \geq 1\). Moreover, \(X_n \to X = 0\) as \(n \to \infty\). However, \(E(X_n) = 1\) does not converge to \(E(X) = 0\) as \(n \to \infty\). You see that this example violates both assumptions of Lebesgue’s theorem.

V. RANDOM WALK

Fact 3: Assume \(P[X_{n+1} = i + 1|X_n = i] = p\) and \(P[X_{n+1} = i - 1|X_n = i] = q = 1 - p\) for \(i \in \mathcal{X} = \mathbb{Z}\). Here, \(p \in (0, 1)\).

(a) If \(p \neq 0.5\), the Markov chain is transient.
(b) If \(p = 0.5\), the Markov chain is null recurrent.

Proof:
(a) Assume \(p < 0.5\). Fix \(a,b \in \{1,2,3,\ldots\}\). Define \(\alpha(i) = P[T_a < T_b|X_0 = i], -b \leq i \leq a\). Then, according to Kolmogorov’s equations,
\[
\alpha(i) = \begin{cases} 
\sum_j P(i,j)\alpha(j), & \text{if } i \notin A \cup B; \\
1, & \text{if } i = a; \\
0, & \text{if } i = -b.
\end{cases}
\]

Solving these equations, assuming \(p < 0.5\), we find
\[
P[T_a < T_b|X_0 = i] = \frac{p^i - p^b}{p^a - p^{-b}}, -b \leq i \leq a \text{ where } \rho := \frac{q}{p} > 1.
\]

Now we let \(b \to \infty\). We use part (c) of Fact 2 to conclude that
\[
P[T_a < \infty|X_0 = i] = \rho^{i-a}, i \leq a.
\]

Since this value is less than 1, we see that
\[
P[T_a < \infty|X_0 = a] = pP[T_a < \infty|X_0 = a+1] + qP[T_a < \infty|X_0 = a-1] < 1,
\]
so that \(a\) is transient.

(b) Assume \(p = 0.5\). In that case, the solution of (1) is
\[
P[T_a < T_b|X_0 = i] = \frac{i + b}{a + b}.
\]

Consequently, as \(b \to \infty\), we find
\[
P[T_a < \infty|X_0 = i] = 1
\]

and we conclude that every state is recurrent. To show positive recurrence, we solve the Kolmogorov equations for \(\beta(i) = E[\min\{T_a, T_b\}|X_0 = i]\). We find
\[
E[\min\{T_a, T_b\}|X_0 = i] = (a - i)(i + b), -b \leq i \leq a.
\]

Letting \(b \to \infty\), we find
\[
E[T_a|X_0 = i] = \infty, i \neq a,
\]
which proves null recurrence.

REFERENCES