EE226a - Summary of Lecture 23
Continuous Time Markov Chains: Key Results

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I. SUMMARY

We explain the following ideas and results:
• Invariant Distribution
• Classification Theorem
• M/M/1 queue
• Time-Reversal

II. INVARIANT DISTRIBUTION

We have the following definition.

Definition 1: Invariant Distribution
The distribution \( \pi = \{ \pi(i), i \in \mathcal{X} \} \) is said to be invariant for \( Q \) if
\[ \pi Q = 0, \]

i.e., if
\[ \pi(i)(-q(i,i)) = \sum_{j \in \mathcal{X}} \pi(j)q(j,i), i \in \mathcal{X}. \] (1)

The relations (1) are called the balance equations.
The relevance is because of the following results.

Theorem 1: Kolmogorov Equations
Let \( \mathcal{X} \) be a regular Markov chain (meaning, no explosions are possible) with rate matrix \( Q \).
For \( t \geq 0 \), let \( \pi_t = \{ \pi_t(i), i \in \mathcal{X} \} \) where \( \pi_t(i) = P(X_t = i) \). We consider \( \pi_t \) as a row vector.
Then
\[ \frac{d}{dt} \pi_t = \pi_t Q. \]

Consequently, \( \pi_t = \pi \) for all \( t \geq 0 \) if and only if \( \pi_0 = \pi \) and \( \pi \) is invariant.

We discuss an interpretation of the balance equations (1). The number of transitions from \( \{i\} \) to \( \{i\}^c := \{j \in \mathcal{X} \mid j \neq i\} \) and the number of transitions from \( \{i\}^c \) to \( \{i\} \) over any interval \([0, T]\) differ by at most one. If the Markov chain is stationary with invariant distribution \( \pi \), then the rates of those transitions are given by both sides of (1).

Theorem 2: Let \( \mathcal{X} \) be a regular Markov chain with rate matrix \( Q \) and initial distribution \( \pi \). It is stationary if and only if \( \pi \) is invariant.

III. CLASSIFICATION THEOREM

We define irreducible, transient, null recurrent, and positive recurrent as in discrete time. We have the following result.

Theorem 3: Classification
Let \( \mathcal{X} = \{X_t, t \geq 0\} \) be an irreducible Markov chain on \( \mathcal{X} \).
(a) The states are either all transient, all null recurrent, or all positive recurrent. We then say that the Markov chain is ....
(b) If \( \mathcal{X} \) is transient or null recurrent, then
\[ \frac{1}{T} \int_0^T 1\{X_t = i\}dt \to 0, \text{ as } T \to \infty, \forall i \in \mathcal{X}, \text{ a.s.} \]
Moreover, there is no invariant distribution and
\[ P(X_t = i) \to 0, \forall i \in \mathcal{X}. \]
(c) If \( \mathcal{X} \) is positive recurrent, then
\[ \frac{1}{T} \int_0^T 1\{X_t = i\}dt \to \pi(i) > 0, \text{ as } T \to \infty, \forall i \in \mathcal{X}, \text{ a.s.} \]
Moreover, π is the unique invariant distribution and

\[ P(X_t = i) \rightarrow \pi(i), \forall i \in \mathcal{X}. \]

**Proof:**

Consider the jump chain \( Y = \{Y_n, n \geq 0\} \) that specifies the sequence of successive values of \( X \). We can see that \( X \) and \( Y \) must be of the same type.

We leave the details as an exercise.

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**IV. M/M/1 Queue**

Customers arrive according to a Poisson process with rate \( \lambda \) and get served one by one, with independent service times that are exponentially distributed with rate \( \mu \), by a single server.

**Theorem 4:** The number of customers in the queue at time \( t \), \( X_t \), is a Markov chain with rate matrix \( q(n, n + 1) = \lambda \), \( q(n + 1, n) = \mu \) for \( n \geq 0 \). All the other non-diagonal terms are zero.

This Markov chain is positive recurrent if and only if \( \lambda < \mu \). In that case, the unique invariant distribution is

\[ \pi(n) = (1 - \rho)\rho^n, n \geq 0. \]

**Proof:**

Write the balance equations. You find \( \pi(n) = \rho^n\pi(0), n \geq 0 \). There is a solution only if \( \rho < 1 \).

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**V. Time Reversal**

We want to examine the Markov chain \( X_t \) observed in reverse time. Imagine a movie of the Markov chain that you play in reverse.

The first observation may be a bit surprising.

**Fact 1:** Assume \( \{X_t, t \geq 0\} \) is a Markov chain. In general, \( \{Y_t := X_{T-t}, 0 \leq t \leq T\} \) is not a Markov chain.

**Proof:**

Exercise.

However, we have the nice result.

**Theorem 5:** Assume \( X \) is a regular stationary Markov chain with rate matrix \( Q \) and invariant distribution \( \pi \). Then \( Y \) is a regular stationary Markov chain with invariant distribution \( \pi \) and rate matrix \( Q' \) where

\[ q'(i, j) = \frac{\pi(j)q(j, i)}{\pi(i)}. \]

**Definition 2:** Time-Reversible A random process \( X \) is time-reversible if ...

**Theorem 6:** A Markov chain with rate matrix \( Q \) is time-reversible if it is stationary and its invariant distribution \( \pi \) is such that

\[ \pi(i)q(i, j) = \pi(j)q(j, i), \forall i, j \in \mathcal{X}. \]  \hspace{1cm} (2)

These relations are called the **detailed balance equations**.

Here is a cute application to the M/M/1 queue.

**Theorem 7:** The stationary M/M/1 chain is time-reversible. In particular, the departures from the stationary queue form a Poisson process with rate \( \lambda \) whose past up to time \( t \) is independent of \( X_t \). It follows that two M/M/1 queue in tandem, when stationary, have independent queue lengths at any given time.

**Proof:**

The stationary distribution satisfies the detailed balance equations (2), so that the stationary queue length process is time-reversible.

The departures up to time \( t \) become the arrivals after time \( t \) for the time-reversed queue. Since the time-reversed queue is again an M/M/1 queue, the arrivals after time \( t \) are a Poisson process independent of \( X_t \). Therefore, the same is true of the departures before time \( t \).

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**REFERENCES**
