I. SUMMARY

Here are the key ideas and results:

- **Theorem 1**: $K$ is a covariance matrix iff it is positive semi-definite; then $K = R^2 = QAQ^T$ for some orthogonal matrix $Q$ and $R = QA^{1/2}Q^T$.
- **Theorem 2**: If $X = N(\mu, K)$ with $|K| \neq 0$, then $f_X = (1)$

II. COVARIANCE MATRICES

Assume that $K$ is a covariance matrix. That means that $K = E(XX^T)$ for some zero-mean random vector $X$. Here are some basic properties.

*Theorem 1*: Properties of Covariance Matrix

Assume that $K$ is a covariance matrix. That matrix must have the following properties.

1. $K$ is positive semi-definite. That is, $a^T K a \geq 0$ for all $a \in \mathbb{R}^n$.
2. $K$ is positive definite if and only if $|K| \neq 0$.
3. The eigenvalues of $K$ are real and nonnegative. Let $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of $K$ repeated according to their multiplicity. There is an orthonormal matrix $Q$ such that $KQ = QA\Lambda$.
4. If $K$ is positive definite, then $K^{-1} = QA^{-1}Q^T$.
5. There is a unique positive semi-definite matrix $R$ such that $K = R^2$ and $R = QA^{1/2}Q^T$.
6. A positive semi-definite symmetric matrix $K$ is a covariance matrix. It is the covariance matrix of $RX$ where $X = N(0, I)$.

*Proof:*

1. Assume $K = E(XX^T)$ for some zero-mean random vector $X$. For $a \in \mathbb{R}^n$ one has $a^T K a = E(Y^2)$ where $Y = a^T X$.

   Hence $a^T \Sigma a \geq 0$.

2.-(4) Since $K$ is positive semi-definite, (2)-(4) follow from Theorem 6 in [2].

5. The issue is uniqueness. The matrix $R$ is such that $R = VA^{1/2}V^T$ where $K = VA\Lambda V^T$. Thus, $V$ are the eigenvectors of $K$ and $A^{1/2}$ is fixed.

6. is immediate.

The above theorem tells us about the shape of $f_X$, as stated in the next result, illustrated in Fig. 1.

*Theorem 2*: Assume that $X = N(0, K)$. If $|K| = 0$, the RVs $X$ do not have a joint density. If $|K| \neq 0$, then

$$f_X(x) = \frac{1}{(2\pi)^{n/2}|K|^{1/2}} \exp\{-\frac{1}{2}x^T K^{-1}x\}. \tag{1}$$

Also, the level curves of $f_X$ are ellipses whose axes are the eigenvectors of $K$ and dimensions scaled by the square roots of the eigenvalues of $K$. 
Fig. 1. The $N(0,K)$ probability density function.

**Proof:**

The expression for $f_X$ follows from the representation $X = RY$ and the observation that if $x = Ry$, then $y = R^{-1}x$ and $y^T y = x^T R^{-2} x = xK^{-1}x$.

The level curves are sets of $x$ such that $x^T K^{-1} x = y^T y = c$ where $x = Ry$. Thus, $y$ belongs to a circle with radius $\sqrt{c}$ and $x$ belongs to an ellipse whose axes are the eigenvectors $u_i$ of $R$. (See Section VIII in [2].)

**References**
