I. Question 1 - Select the true statements

Two random variables are independent if

[Recall the definition: X and Y are independent if \( P(X \in A \text{ and } Y \in B) = P(X \in A)P(Y \in B), \forall A, B \in B(\mathbb{R}). \)]

(a) They are uncorrelated
This is false. You have to provide a counterexample in H1.

(b) Their joint density is the product of their individual densities
This is true. In that case,

\[
P(X \in A \text{ and } Y \in B) = \int_A \int_B f_{X,Y}(x,y)\,dx\,dy
\]

\[
= \int_A \left[ \int_B f_Y(y)\,dy \right] f_X(x)\,dx \quad \text{[by Fubini]}
\]

\[
= \int_A P(Y \in B) f_X(x)\,dx = P(X \in A)P(Y \in B).
\]

(c) Their joint cdf is the product of their cdf
This is true. Assume

\[
F_{X,Y}(x,y) = F_X(x)F_Y(y).
\]

Then

\[
P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y).
\]

It follows that

\[
P(X \in (a,b], Y \leq y)
= P(X \leq b, Y \leq y) - P(X \leq a, Y \leq y)
= P(X \leq b)P(Y \leq y) - P(X \leq a)P(Y \leq y)
= [P(X \leq b) - P(X \leq a)]P(Y \leq y)
= P(X \in (a,b])P(Y \leq y).
\]

Subtracting this identity with \( y = c \) from the same identity with \( y = d \), we can see that

\[
P(X \in (a,b], Y \in (c,d]) = P(X \in (a,b])P(Y \in (c,d]).
\]

Now taking the sum of this identity over a countable collection of disjoint intervals \( (a_i, b_i) \) whose union is \( A \) and a countable collection of disjoint intervals \( (c_j, d_j) \) whose union is \( B \), we find

\[
P(X \in A, Y \in B) = \sum_{i} \sum_{j} P(X \in (a_i, b_i], Y \in (c_j, d_j])
\]

\[
= \sum_{i} \sum_{j} P(X \in (a_i, b_i])P(Y \in (c_j, d_j])
\]

\[
= \sum_{i} \left[ \sum_{j} P(Y \in (c_j, d_j]) \right] P(X \in (a_i, b_i])
\]

\[
= P(X \in A)P(Y \in B).
\]

Hence, \( P(X \in A, Y \in B) = P(X \in A)P(Y \in B), \forall A, B \in B(\mathbb{R}) \), so that \( X \) and \( Y \) are independent.

(e) Each random variable is a function of a distinct independent random variable
This is true. Say that \( X = f(V) \) and \( Y = g(W) \) where \( V, W \) are independent. We claim that \( X \) and \( Y \) are independent. To show that, we must prove that \( P(X \in A, Y \in B) = P(X \in A)P(Y \in B), \forall \ldots \). The key idea is that, if \( h(\cdot) \) is a function and \( A \) a set in the range of that function, then

\[
h(u) \in A \text{ iff } u \in h^{-1}(A) := \{ v \mid h(v) \in A \}.
\]

Now,

\[
P(X \in A, Y \in B) = P(f(V) \in A, g(W) \in B)
\]

\[
= P(V \in f^{-1}(A), W \in g^{-1}(B))
\]

\[
= P(V \in f^{-1}(A))P(W \in g^{-1}(B)) \quad \text{since } V, W \text{ ind.}
\]

\[
= P(f(V) \in A)P(g(W) \in B) = P(X \in A)P(Y \in B).
\]

(d) They are indicators of independent sets.
This is true. We say that \( X \) is the indicator of a set \( A \) if \( X(\omega) = 1 \{ \omega \in A \} \). That is, \( X(\omega) = 1 \) if \( \omega \in A \) and \( X(\omega) = 0 \) if \( \omega \notin A \).

Assume then that \( X(\omega) = 1 \{ \omega \in A \} \) and \( Y(\omega) = 1 \{ \omega \in B \} \) where \( A \) and \( B \) are independent sets, i.e., \( P(A \cap B) = P(A)P(B) \). We must check that

\[
P(X \in C, Y \in D) = P(X \in C)P(Y \in D), \forall C, D \ldots
\]

Since \( X \) and \( Y \) take only the values 0 and 1, there are only finitely many possibilities. Say that \( 0 \notin C, 1 \in C, 0 \notin D, 1 \in D \). Then

\[
P(X \in C, Y \in D) = P(X = 1, Y = 1) = P(A \cap B)
\]

\[
= P(A)P(B) = P(X = 1)P(Y = 1) = P(X \in C)P(Y \in D).
\]

The other cases are similar.

(e) One is the sum, the other the difference of two independent random variables.
This is false. For instance, let \( V = U[0,1] \) and \( W = 0 \). These are two independent random variables. Indeed, a constant random variable is independent of any other random variable as you can see directly from the definition of independence. Now,

\[
X = V + W = V \quad \text{and} \quad Y = V - W = V
\]

are certainly not independent.

II. Question 2: Simple Facts About Probability - True or False

1. **A random variable is any function of \( \omega \).**
   This is false. Assume that \( \Omega = [0,1] \) and \( \mathcal{F} = \{\emptyset, [0,0.5], (0.5,1], [0,1]\}. You can check that the collection \( \mathcal{F} \) is closed under countable set operations. Now assume that \( X(\omega) = \omega \). Then \( X^{-1}(\{0,0.3\}) = [0,0.3] \notin \mathcal{F} \). That is, the inverse image of some interval is not in \( \mathcal{F} \) and \( X \) is not a random variable.

   The point of this example is that you might know that \( P([0,0.5]) = 0.32 \), which tells you the probability of the other sets in \( \mathcal{F} \). This probability space does not specify \( P(X \leq 0.3) \). Thus, the probability space is not 'rich' enough to 'measure' \( X \).

2. **\( E[X|Y] \) is a random variable**
   This is true. By definition, \( E[X|Y] \) is a function of \( Y \) and is a random variable. For instance, assume that \( \omega \) is picked uniformly in the triangle \( \Omega = (\omega = (\omega_1, \omega_2) | \omega_1 \in [0,1], \omega_2 \in [0,\omega_1]) \). Then (draw a picture),

\[
E[X|Y] = (1 + Y)/2
\]

You will agree that \((1 + Y)/2\) is a random variable.

3. **\( E[X|Y] = E[Y|X] \)**
   This is false. In the example of (2), \( E[X|Y] = (1 + Y)/2 \) and \( E[Y|X] = X/2 \).

   This is false. Let \( \{\Omega, \mathcal{F}, P\} \) be the uniform probability on \([0,1]\). With \( A = [0,0.2] \) and \( B = [0.1,1] \), we find

\[
P[A|B] = \frac{0.1}{0.9} \neq P[B|A] = \frac{0.1}{0.2}.
\]

5. **\( E(X^2) \geq E(X)^2 \)**
   This is true. The inequality follows from the fact that \( f(x) = x^2 \) is convex and, for any convex function \( f(\cdot) \), one has \( E(f(X)) \geq f(E(X)) \). This is Jensen's inequality.

   For this particular function, one can also argue that

\[
0 \leq \text{var}(X) = \text{E}((X - E(X))^2) = E(X^2) - (E(X))^2.
\]

6. **\( P((a,b]) = \lim_{n \to \infty} P((a,b-1/n]) \)**
   This is false. What is true is that

\[
P((a,b]) = \lim_{n \to \infty} P((a,b-1/n]).
\]