EE226A: Random Processes in Systems

Problem Set 11

Due: Tuesday, December 13 in 264M between 1:00 pm and 3:00 pm

1. Suppose \((X_n, n \geq 1)\) is a sequence of (not necessarily independent) r.v’s. Assume \(\epsilon_n \downarrow 0\) and
\[ P(|X_n| \geq \epsilon) \leq \frac{1}{n^2}, \forall n \geq 1 \text{ and } \forall \epsilon > 0. \]

Show \(X_n \xrightarrow{a.s.} 0\), as \(n \to \infty\).

[Hint: Use Borel-Cantelli lemma.]

2. Assume \(\epsilon_n > 0\) are such that \(\sum_{n=1}^{\infty} \epsilon_n < \infty\) and
\[ P(|X_k - X_m| > \epsilon_n) \leq 2^{-n}, \text{ for all } k, m \geq n \geq 1. \]

Show that \((X_n, n \geq 1)\) is Cauchy, a.s., and conclude that \(X_n \xrightarrow{a.s.} X\), for some finite r.v. \(X\).

[Hint: Use Borel-Cantelli lemma.]

3. Assume
\[ \sup_{k, m \geq n} P(|X_k - X_m| > \epsilon) \to 0, \text{ as } n \to \infty, \forall \epsilon > 0. \]

Show that \(X_n \xrightarrow{P} X\), for some finite r.v. \(X\).

[Hint: Choose \(\epsilon_i\) as in previous problem, and define \(n_i\) such that
\[ P(|X_k - X_m| > \epsilon_i) \leq 2^{-i}, \text{ for all } k, m \geq n_i. \]

Use the previous problem to conclude \(X_{n_i} \xrightarrow{a.s.} X\), for some finite r.v. \(X\). Conclude that \(X_{n_i} \xrightarrow{P} X\).]

4. Construct an example of random variables \((X_n, n \geq 1)\), such that
\[ P(|X_n - X_m| > \epsilon) \leq 0.01, \text{ for all } n, m, \]

but
\[ P(\sup_{n \neq m} |X_n - X_m| > \epsilon) = 1. \]

5. Let \(X_k\) be a positive recurrent discrete-time MC, and \(f(\cdot)\) any bounded function. Assume \(Y\) is distributed according to the invariant distribution of the MC.

Use SLLN to show
\[ \frac{1}{n} \sum_{k=1}^{n} f(X_k) \xrightarrow{a.s.} E(f(Y)). \]
6. Consider a renewal process with events at times $T_0, T_1, \ldots$, i.e., $(T_n - T_{n-1}, n \geq 1)$ are i.i.d. Assume $T_0 = 0$, and define $X_t = \min\{T_k - t | k \geq 0, T_k - t \geq 0\}$ the residual time for the first event after time $t > 0$.

(a) Show $\frac{1}{T} \int_0^T X_t dt \xrightarrow{a.s.} E((T_1 - T_0)^2) / 2E(T_1 - T_0)$, as $T \to \infty$.

(b) Let $N_t$ be the number of events up to including time $t \geq 0$, i.e., $N_t = \min\{n \geq 1 : T_{n-1} \geq t\}$.

Show $\frac{N_t}{t} \xrightarrow{a.s.} \frac{1}{E(T_1 - T_0)}$.

7. Let $X_t$ be a positive recurrent CTMC with invariant distribution $\pi$. The chain $X_t$ modulates the arrivals of some counting process $N_t$ as follows: while $X_t = i$, $N_t$ increases according to an independent Poisson process of rate $\lambda(i)$. Assume $\max_i \lambda(i) < \infty$.

Show that $\frac{N_t}{t} \xrightarrow{a.s.} \lambda$, where $\lambda = \sum_i \lambda(i) \pi(i)$. 