1. Let us define the r.v’s $U := X - \mu_X$ and $V := Y - \mu_Y$. Then $(U, V)$ is zero-mean JG with variances $\sigma_X^2, \sigma_Y^2$, and correlation coefficient $\rho$. From equation (2.24) of the reader,

$$f_{U|V}(u|v) = \frac{1}{\sigma_X \sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(u - \rho(\sigma_X/\sigma_Y)u)^2}{2\sigma_X^2(1-\rho^2)}\right).$$

But $\{Y = y\} = \{V = y - \mu_Y\}$. Therefore,

$$f_{U|V}(u|y) = \frac{1}{\sigma_X \sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(u - \rho(\sigma_X/\sigma_Y)(y - \mu_Y))^2}{2\sigma_X^2(1-\rho^2)}\right).$$

Since $X = g(U) = U + \mu_Y$, we can use the transformation formula

$$f_X(x|y) = \frac{f_{U|Y}(g^{-1}(x)|y)}{|\frac{dg^{-1}(x)}{dx}|} = \frac{1}{\sigma_X \sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(x - \sigma_X/\sigma_Y)(y - \mu_Y)^2}{2\sigma_X^2(1-\rho^2)}\right).$$

2. (a) The conditional density of $X$ given $Y$ is provided in equation (2.24) of the reader,

$$f_X(x|u) = f_X(x|\sqrt{u})$$

$$f_X(x|v) = \frac{1}{\sigma_X \sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(x - \rho(\sigma_X/\sigma_Y)\sqrt{u})^2}{2\sigma_X^2(1-\rho^2)}\right).$$

(b) This part is harder because the event $\{U = u\}$ does not equal $\{Y = y\}$ for any $y$. However we can apply the technique used in part 2a if we first condition on the sign of $Y$:

$$f_{X|U}(x|u) = f_{X|U}(x|u, Y > 0)P[Y > 0|U = u] + f_{X|U}(x|u, Y \leq 0)P[Y \leq 0|U = u].$$

The event $\{Y > 0\}$ is independent of $U$, so $P[Y > 0|U = u] = P(Y > 0) = 1/2$. For $u > 0$, the events $\{U = u, Y > 0\}$ and $\{Y = \sqrt{u}\}$ are the same, so

$$f_{X|U}(x|u, Y > 0) = \frac{1}{\sigma_X \sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(x - \rho(\sigma_X/\sigma_Y)\sqrt{u})^2}{2\sigma_X^2(1-\rho^2)}\right).$$

Similarly, for $u \geq 0$, the events $\{U = u, Y \leq 0\}$ and $\{Y = -\sqrt{u}\}$ are the same, so

$$f_{X|U}(x|u, Y \leq 0) = \frac{1}{\sigma_X \sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(x + \rho(\sigma_X/\sigma_Y)\sqrt{u})^2}{2\sigma_X^2(1-\rho^2)}\right).$$

Thus,

$$f_{X|U}(x|u) = \frac{1}{\sigma_X \sqrt{2\pi(1-\rho^2)}} \left(\exp\left(-\frac{(x - \rho(\sigma_X/\sigma_Y)\sqrt{u})^2}{2\sigma_X^2(1-\rho^2)}\right) + \exp\left(-\frac{(x + \rho(\sigma_X/\sigma_Y)\sqrt{u})^2}{2\sigma_X^2(1-\rho^2)}\right)\right).$$
3. Notice that $X = AB^{-1}Y$. Thus, $P[X = z|Y] = 1\{z = AB^{-1}Y\}$. Hence, $f_{X|Y}(\cdot|\cdot)$ does not exist.

If you like using impulse functions, then pdfs always exist, so in this case:

$$f_{X|Y}(x|y) = \delta(AB^{-1}y - x),$$

where $\delta(u) = 0$ for all $u \neq 0$ and $\int_{-\infty}^{\infty} \delta(u)du = 1$.

4. Look at the $K_{Y_1,Y_2}$ submatrix of $K$, and observe that $E(Y_1Y_2) = \sqrt{E(Y_1^2)E(Y_2^2)}$. So, by Cauchy-Schwartz, $Y_1 = 2Y_2$. This suggests that if we set

$$\alpha = \left(\begin{array}{c} 2 \\ 1 \end{array} \right),$$

and $Z = Y_2$, then $Y = AZ, K_Z = 1$, with $(X,Z) = (X,Y_2)$ being JG. Since $|K_{X,Z}| = |K_{X,Y_2}| = 3 \neq 0$, we can apply (2.24) from reader

$$f_{X|Z}(x,z) = f_{X|Y_2}(x,z) = \frac{1}{\sqrt{6\pi}} \exp\left(-\frac{(x-z)^2}{6}\right).$$

5. Define $Y'_2 := (Y_1 - Y_3)/2$. Then,

$$\left( \begin{array}{c} Y_1 \\ Y_2' \\ Y_3 \end{array} \right) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 1 & 0 \end{array} \right) \left( \begin{array}{c} Y_1 \\ Y_3 \end{array} \right).$$

Then, $(Y_1, Y_2', Y_3)$ is JG, and it is easy to check that $K_{Y_1,Y_2',Y_3} = K$.

If we show that $Y'_2 = Y_2$, then we are done since then we can define

$$A = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 1 & 0 \end{array} \right),$$

and $Z = \left( \begin{array}{c} Y_1 \\ Y_3 \end{array} \right)$.

Notice that $E(Y_2Y'_2) = E(Y_2(Y_1 - Y_3)/2) = 2, E(Y'_2^2) = 2, E(Y_2^2) = 2$, so

$$E(Y_2Y'_2)^2 = E(Y'_2^2)E(Y_2^2).$$

By the Cauchy-Schwartz inequality, $Y_2 = \alpha Y'_2$ for some $\alpha \in \mathbb{R}$. But, $E(Y_2Y'_2) = \alpha E(Y'_2^2)$, so $\alpha = 1$.

Another solution, is to solve

$$\left( \begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \end{array} \right) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 1 & 0 \end{array} \right) \left( \begin{array}{c} Z_1 \\ Z_2 \end{array} \right),$$

w.r.t. $(Z_1, Z_2)$:

Now,

$$\left( \begin{array}{ccc} 1 & 0 & 0 \\ 1 & -2 & 0 \end{array} \right) \left( \begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \end{array} \right) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 1 & -2 & 0 \end{array} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 1 & 0 \end{array} \right) \left( \begin{array}{c} Z_1 \\ Z_2 \end{array} \right),$$

i.e.,

$$\left( \begin{array}{c} Y_1 \\ Y_1 - 2Y_2 \end{array} \right) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} Z_1 \\ Z_2 \end{array} \right),$$

so $Z_1 = Y_1, Z_2 = Y_1 - 2Y_2$.

Yet, another approach would be to follow the hint given.
6. \( K_Y = QAQ^T \), for some orthogonal \( Q \), diagonal \( \Lambda \), where

\[
\Lambda = \begin{pmatrix} \Lambda_m & 0 \\ 0 & 0 \end{pmatrix},
\]

for some \( m < n, \Lambda_m \in \mathbb{R}^{m \times m}, |\Lambda_m| > 0 \).

Set

\[
A = Q \begin{pmatrix} \Lambda_m^{1/2} \\ 0 \end{pmatrix}.
\]

We will solve \( Y = AZ \) w.r.t. \( Z \in \mathbb{R}^m \):

\[
Y = Q \begin{pmatrix} \Lambda_m^{1/2} \\ 0 \end{pmatrix} Z,
\]

\[
Q^T Y = \begin{pmatrix} \Lambda_m^{1/2} \\ 0 \end{pmatrix} Z.
\]

\[
(\Lambda_m^{-1/2} \ 0)Q^T Y = (\Lambda_m^{-1/2} \ 0) \begin{pmatrix} \Lambda_m^{1/2} \\ 0 \end{pmatrix} Z = I_m Z
\]

\[
(\Lambda_m^{-1/2} \ 0)Q^T Y = Z.
\]

By defining \( B := (\Lambda_m^{-1/2} \ 0)Q^T \), \( Z := BY \), we have \( Y = AZ \) by the above. Also,

\[
K_Z = E(BYY^T B^T) = (\Lambda_m^{-1/2} \ 0)Q^T QAQ^T \begin{pmatrix} \Lambda_m^{-1/2} \\ 0 \end{pmatrix} = I_m.
\]

That \( (X, Z) \) is JG, is obvious.

We know, \( D = K_{xz}K_Z^{-1} = E(XY^T B^T) = K_{xy}B^T \) and \( \Sigma = K_{xx} - K_{xz}K_{z}^{-1}K_{zx} = K_{xx} - K_{xy}B^T BK_{yx} \).

By the bijection between \( Y \) and \( Z \) established above, \( \{Y = y\} = \{Z = By\} \). But, given \( Z = BY, X = N(DBY, \Sigma) \). Thus, \( M = DB, S = \Sigma \).