Solutions to Problem Set 4

1. Given some observation \( y = (y_1, \ldots, y_n) \in \{0, 1\}^n \), the likelihood ratio \( \Lambda(y) \) is

\[
\Lambda(y) = \frac{P[Y = y \mid X = 1]}{P[Y = y \mid X = 0]} = \frac{\sum_{i=1}^n y_i (1 - p_i)^{n - \sum_{i=1}^n y_i}}{p_0^{\sum_{i=1}^n y_i} (1 - p_0)^{n - \sum_{i=1}^n y_i}}. \tag{1}
\]

The Neyman-Pearson test is to choose \( Z = 1 \) with probability \( \phi(y) \) given by

\[
\phi(y) = \begin{cases} 
1, & \text{if } \Lambda(y) > \lambda \\
\gamma, & \text{if } \Lambda(y) = \lambda \\
0, & \text{otherwise,}
\end{cases}
\]

for \( \lambda, \gamma \) given by

\[
P[Z = 1 \mid X = 0] = \beta \iff P[\Lambda(Y) > \lambda \mid X = 0] + \gamma P[\Lambda(Y) = \lambda \mid X = 0] = \beta.
\]

From (1) we see that \( \sum_{i=1}^n Y_i \) is a sufficient statistic.

2. Let \( 3 \geq y_i \geq 0 \) for all \( i = 1, \ldots, n \). Then,

\[
\Lambda(y) = \frac{f_{Y \mid X}(y \mid 1)}{f_{Y \mid X}(y \mid 0)} = \frac{3^{-n}}{2^{-n} 1_{\{\max_i y_i \leq 2\}}} = \begin{cases} 
\frac{(2/3)^n}{\max_i y_i \leq 2}, & \text{if } \max_i y_i \leq 2 \\
\infty, & \text{otherwise}.
\end{cases} \tag{2}
\]

Set \( \lambda = (2/3)^n \), and \( \gamma = \beta \). Then,

\[
P[Z = 1 \mid X = 0] = P[\Lambda(Y) > \lambda \mid X = 0] + \gamma P[\Lambda(Y) = \lambda \mid X = 0] = 0 + \gamma 1 = \beta.
\]

From (2) we see that \( \max_i Y_i \) is a sufficient statistic.

3. \( \Lambda(y) = \frac{f_{Y \mid H}(y \mid 1)}{f_{Y \mid H}(y \mid 0)} = \frac{1}{2^n} \prod_{i=1}^n \exp((3y_i^2 - 8y_i + 4)/8). \tag{3} \]

Since \( \Lambda(y) \) is a continuous r.v., \( \gamma \) is not needed, and \( \lambda \) is (uniquely) determined by

\[
P[\Lambda(Y) > \lambda_0 \mid X = 0] = \beta.
\]

From (3) we see that \( \sum_i (3Y_i^2 - 8Y_i) \) is a sufficient statistic.

4. (a)

\[
\Lambda(y) = \frac{f_{Y \mid H}(y \mid 1)}{f_{Y \mid H}(y \mid 0)} = \exp \left( \frac{y_1^2 + y_2^2}{2(\sigma^2 + a^2)} + \frac{y_3^2 + y_4^2}{2\sigma^2} - \frac{y_5^2 + y_6^2}{2(\sigma^2 + a^2)} \right).
\]

Thus, \( \Lambda(y) > 1 \iff y_1^2 + y_2^2 > y_3^2 + y_4^2 \). 1
(b) By Problem 4 in HW#1 we know that given $H$, $V_0, V_1$ are exponential r.v.'s. In particular, 

$$f_{V_0|H}(v_0|0) = \frac{1}{2(\sigma^2 + a^2)} \exp \left( -\frac{v_0}{2(\sigma^2 + a^2)} \right), \quad f_{V_1|H}(v_1|0) = \frac{1}{2\sigma} \exp \left( -\frac{v_1}{2\sigma} \right).$$

(c) Fix $u \in \mathbb{R}$. Using the fact that conditional on $H$, $V_0, V_1$ are independent exponential r.v's,

$$P[V_0 - V_1 > u|H = 0] = P[V_0 > V_1 + u|V_0 > V_1, H = 0]P[V_0 > V_1|H = 0]$$

$$+ (1 - P[V_1 \geq V_0 - u|V_0 \geq V_1, H = 0])P[V_1 \geq V_0|H = 0]$$

$$= P[V_0 > u|H = 0]P[V_0 > V_1|H = 0]$$

$$+ (1 - P[V_1 \geq -u|H = 0])P[V_1 \geq V_0|H = 0]$$

$$= e^{-\frac{u}{\sigma^2 + a^2}} \frac{2(\sigma^2 + a^2)}{2(\sigma^2 + a^2) + 2\sigma^2} \{u > 0\} + \frac{2(\sigma^2 + a^2)}{2(\sigma^2 + a^2) + 2\sigma^2} \{u \leq 0\}$$

$$+ 1\{u \leq 0\}(1 - e^{-\frac{u}{\sigma^2 + a^2}}) \frac{2\sigma^2}{2(\sigma^2 + a^2) + 2\sigma^2} \{u \leq 0\},$$

so

$$f_U|H(u|0) = \frac{d}{du} (1 - P[U > u|H = 0]) = \begin{cases} \frac{1}{2(\sigma^2 + a^2) + 2\sigma^2} e^{-\frac{u}{\sigma^2 + a^2}}, & \text{if } u > 0 \\ \frac{1}{2(\sigma^2 + a^2) + 2\sigma^2} e^{\frac{u}{\sigma^2 + a^2}}, & \text{if } u \leq 0. \end{cases}$$

(d) 

$$P[e|H = 0] = P[V_0 < V_1|H = 0] = \frac{2\sigma^2}{2(\sigma^2 + a^2) + 2\sigma^2}.$$

5. (a) We will use (3.15) from Gallager’s notes. Substituting $a = 5$ and $b = 1$ and rearranging, we see that the decision is $\hat{H} = 1$ if

$$Y_1 \leq \frac{-\sigma^2 \log(P_0/P_1)}{4} + 3,$$

otherwise $\hat{H} = 0$. Following the exposition leading up to (3.18) and (3.19) in the notes (which assumes $b > a$), we see that

$$P[e|H = 0] = Q \left( \frac{\sigma \log(P_0/P_1)}{4} + \frac{2}{\sigma} \right),$$

$$P[e|H = 1] = Q \left( -\frac{\sigma \log(P_0/P_1)}{4} + \frac{2}{\sigma} \right).$$

(b) The MAP rule is: choose $\hat{H} = 1$ if

$$P[H = 1|Y_1 = y_1, Y_2 = y_2] \geq P[H = 0|Y_1 = y_1, Y_2 = y_2],$$

and $\hat{H} = 0$ otherwise. But this is equivalent to each of the following:

$$f_{Y_1, Y_2|H}(y_1, y_2|1)P_1 \geq f_{Y_1, Y_2|H}(y_1, y_2|0)P_0,$$

$$f_{Y_1, Y_2|H}(y_2|y_1, 1)f_{Y_1|H}(y_1|1)P_1 \geq f_{Y_1, Y_2|H}(y_2|y_1, 0)f_{Y_1|H}(y_1|0)P_0. \quad (4)$$

But $Y_2$ and $H$ are conditionally independent given $Y_1$: $f_{Y_1, Y_2|H}(y_2|y_1, 0) = f_{Z_2}(y_2 - y_1)$, $f_{Y_1, Y_2|H}(y_2|y_1, 1) = f_{Z_2}(y_2 - y_1)$. Thus the first two factors in (4) cancel from both sides of the inequality, and we see that $Y_1$ is a sufficient statistic. Thus, the MAP rule and the probabilities of error are the same as in the previous part.

(c) The researcher introduced $Y_2$, which is just a noisy version of $Y_1$. But (a clean version of) $Y_1$ is already available, so $Y_2$ provides no new information about the disease.
The solution in (b) does not use the fact that $Z_2$ is Gaussian, only that $Y_2$ and $H$ are conditionally independent given $Y_1$. So the solution is the same as in (b).

By (d), we can disregard $Y_2$ and just redo (a) with $Z_1 = U[0,1]$. Let $u(\cdot)$ be the unit-step function. The MAP rule is to choose $\hat{H} = 1$ if

$$P[H = 1|Y_1 = y_1] \geq P[H = 0|Y_1 = y_1] \Leftrightarrow f_{Y_1|H}(y_1|1)P_1 \geq f_{Y_1|H}(y_1|0)P_0 \Leftrightarrow (u(y_1 - 1) - u(y_1 - 2))P_1 \geq (u(y_1 - 5) - u(y_1 - 6))P_0.$$ 

So the MAP rule is to choose $\hat{H} = 1$ if $1 \leq y_1 \leq 2$ and $\hat{H} = 0$ if $5 \leq y_1 \leq 6$. Note that $Y_1$ will always fall in one of these two regions so our rule is completely determined.

The probability of error is 0, since $P[Y_1 \leq 2|H = 0] = 0$ and $P[Y_1 \geq 5|H = 1] = 0$.

6. (a)

$$L[\cos(U)|U + V] = \frac{\text{Cov}(\cos(U), U + V)}{\text{Var}(U + V)}(U + V) + E(\cos(U)) - \frac{\text{Cov}(\cos(U), U + V)}{\text{Var}(U + V)}E(U + V)$$

Now,

$$\text{Cov}(\cos(U), U + V) = \text{Cov}(\cos(U), U) + \text{Cov}(\cos(U), V) = E(U\cos(U)) - E(U)E(\cos(U)),$$

by independence of $U, V$. But,

$$E(U\cos(U)) = \int_0^1 u \cos(u)du = \sin(1) - \int_0^1 \sin(u)du = \sin(1) + \cos(1) - 1,$$

so $\text{Cov}(\cos(U), U + V) = \cos(1) + 0.5\sin(1) - 1$. Also, $\text{Var}(U + V) = \text{Var}(U) + \text{Var}(V) = 1/6$, and

$$E(\cos(U)) = \int_0^1 \cos(u)du = \sin(1).$$

(b) Notice that $(U, V)$ is uniformly distributed on the unit square. On $U + V = s$, $(U, V) = (U, s - U)$ is uniformly distributed, so $U$ is $U[0,s]$ if $s < 1$, and $U[s-1,1]$ if $s \geq 1$. Hence, if $s \geq 1$,

$$E[\cos(U)|U + V] = \int_{s-1}^1 \frac{\cos(u)}{2-s}du = \frac{\sin(1) - \sin(s-1)}{2-s},$$

and

$$E[\cos(U)|U + V] = \int_0^s \frac{\cos(u)}{s}du = \frac{\sin s}{s},$$

if $s < 1$.

(c) From theorem proved in class, we expect $E((X - \cos(U))^2) \geq E((Y - \cos(U))^2)$.