1. [Strong Markov Property.] Consider a MC \((X_n, n \geq 0)\) with transition matrix \(P\). Let \(T_i = \min\{n \geq 0|X_n = i\}\), and assume \(P[T_i < \infty|X_0 = k] = 1\), for all \(i, k\).

Show that \(P(X_{T_i+1} = j) = P(i, j)\).

Conclude that the times between successive returns to \(i\) are i.i.d.

2. Let \(\tau_1, \tau_2, \ldots\) be i.i.d. \(\text{Exp}(\lambda)\) r.v.'s, independent of a geometric r.v. \(Z\) with parameter \(p\), \(0 < p < 1\), i.e., \(P(Z = k) = (1-p)^{k-1}p\), for \(k \geq 1\).

Show that \(\sum_{i=1}^{Z} \tau_i \overset{\text{D}}{=} \text{Exp}(\lambda p)\).

3. Let \(\tau_1, \ldots, \tau_n\) be \(n\) i.i.d. \(\text{Exp}(\lambda)\) r.v.'s.

Calculate \(E(\max\{\tau_1, \ldots, \tau_n\})\).

4. Let \(N\) be a Poisson r.v. with mean \(\lambda\), and \(M\) a binomial r.v. \(B(m, p)\), with \(p = 1 - e^{-\lambda/m}, m > 0\) i.e.,

\[P(M = n) = \binom{m}{n}p^n(1-p)^{m-n}, \text{ for all } n = 0, \ldots, m.\]

Show that \(P(N \leq n) \leq P(M \leq n)\), for any \(n \geq 0\).

(Hint: write \(N = N(1) = \sum_{i=1}^{m}(N(i/m) - N((i-1)/m))\), where \(N(t)\) is a Poisson process with rate \(\lambda\). Define a r.v. \(M\) for which \(N \geq M,\) and \(M \overset{\text{D}}{=} B(m, p)\).)

5. In this problem we want to estimate the rate \(\lambda\) of some Poisson process \(N_t\), given the observation \((N_s: 0 \leq s \leq t)\), for some fixed \(t > 0\).

(a) Show that \(N_t/t\) is the maximum-likelihood estimate of \(\lambda\).

(b) Compute \(\text{Var}(N_t/t)\), for \(t > 0\).

6. Under hypothesis \(X = 0\), \(N_t\) is a Poisson process with rate \(\lambda_0\); under \(X = 1\), \(N_t\) is a Poisson process with rate \(\lambda_1\). Suppose we observe \((N_s: 0 \leq s \leq t)\), for some fixed \(t > 0\).

(a) Solve the hypothesis-testing problem, i.e., find a test that minimizes \(P[\hat{X} = 1|X = 0]\), such that \(P[\hat{X} = 0|X = 1] \leq \beta\).

(b) Find \(t\) such that \(P[\hat{X} = 1|X = 0] \leq \alpha\).
7. [Two-sided stationary Poisson process.] Let \( N_t^+, N_t^- \) be two independent Poisson processes with rate \( \lambda \), and corresponding arrival times \( (T_n^+, n \geq 1), (T_n^-, n \geq 1) \) respectively. For \( n > 0 \), define \( T_n = T_n^+, T_{1-n} = -T_n^- \).

One can think of \( (T_n, n \leq 0) \), \( (T_n, n > 0) \) as describing arrivals occurring before and after time 0, respectively. You will show that the arrival statistics do not depend on the choice of \( t = 0 \), i.e., \( (T_n, n \in \mathbb{Z}) \) is stationary.

For any fixed \( s \in \mathbb{R} \), let \( n_0 = \max\{n \in \mathbb{Z} : T_n \leq s\} \), and consider the sequence \( (T_{n+n_0} - s, n \in \mathbb{Z}) \).

Show that \( (T_{n+n_0} - s, n \in \mathbb{Z}) =_D (T_n, n \in \mathbb{Z}) \).

Observe \( E(T_1 - T_0) = 2\lambda^{-1} \), while \( E(T_{n+1} - T_n) = \lambda^{-1} \) for all \( n \neq 0 \! \) How do you explain this?