Solutions to Problem Set 8

1. 
\[ P(X_{T_i+1} = j) = P(X_{T_i+1} = j, T_i < \infty) \]
\[ = \sum_{t=0}^{\infty} P(X_{t+1} = j, T_i = t) \]
\[ = \sum_{t=0}^{\infty} P(X_{t+1} = j | X_t = i, X_s \neq i \forall s = 0, \ldots, t-1) P(T_i = t) \]
\[ = \sum_{t=0}^{\infty} P(X_{t+1} = j | X_t = i) P(T_i = t) \]
\[ = P(i, j) \sum_{t=0}^{\infty} P(T_i = t) \]
\[ = P(i, j). \]

2. Let \( N_t, t \geq 0 \) be a Poisson process with rate \( \lambda \), and interarrival times \( \tau_1, \tau_2, \ldots \). Then, for all \( s, t \geq 0 \),
\[ P\left[ Z \sum_{i=1}^{Z} \tau_i \geq t + s \right] = P\left[ Z \geq (N_{t+s} - N_s) + N_s | Z \geq N_s \right] \]
\[ = P(Z \geq N_{t+s} - N_s) = P(Z \geq N_t) \]
\[ = P\left( \sum_{i=1}^{Z} \tau_i \geq t \right), \]
so \( \sum_{i=1}^{Z} \tau_i \) has the memoryless property, so it is exponentially distributed with mean
\[ E \left( \sum_{i=1}^{Z} \tau_i \right) = E(Z) E(\tau_i) = p^{-1} \lambda^{-1}. \]

3. Let \( n > 1 \). Then,
\[ E(\max\{\tau_1, \ldots, \tau_n\}) = E(\max\{\tau_1, \ldots, \tau_{n-1}\}) + \]
\[ E(\max\{\tau_1, \ldots, \tau_n\} - \max\{\tau_1, \ldots, \tau_{n-1}\}) \]
But,
\[ E(\max\{\tau_1, \ldots, \tau_n\} - \max\{\tau_1, \ldots, \tau_{n-1}\}) = E((\tau_n - \max\{\tau_1, \ldots, \tau_{n-1}\}) I\{\tau_n > \max\{\tau_1, \ldots, \tau_n\}\}), \]
and
\[ P(\tau_n \geq \max\{\tau_1, \ldots, \tau_{n-1}\} + t | \tau_n \geq \max\{\tau_1, \ldots, \tau_{n-1}\}) = P(\tau_n \geq t), \]
so
\[ E[\max\{\tau_1, \ldots, \tau_n\} - \max\{\tau_1, \ldots, \tau_{n-1}\}|\tau_n > \max\{\tau_1, \ldots, \tau_n\}] = \frac{1}{\lambda}. \]

Hence,
\[ E(\max\{\tau_1, \ldots, \tau_n\}) = E(\max\{\tau_1, \ldots, \tau_{n-1}\}) + \frac{1}{\lambda n} = \frac{1}{\lambda} \sum_{i=1}^{n} \frac{1}{i}. \]

4. Let \( N(t) \) be a Poisson process with rate \( \lambda \), and let \( N = N(1) \). Observe that for any \( m > 0 \),
\[ N = \sum_{i=1}^{m} (N(i/m) - N((i-1)/m)) \geq \sum_{i=1}^{m} 1\{N(i/m) - N((i-1)/m) > 0\} =: M. \]

Since the r.v’s \( N(i/m) - N((i-1)/m), i = 1, \ldots, m \) are independent, \( M = D_B(m, p) \), with
\[ p = P(N(i/m) - N((i-1)/m) = 0) = 1 - e^{-\lambda/m}. \]
Thus, for all \( n \geq 0 \),
\[ P(N \leq n) \leq P(M \leq n). \]

5. (a) Let \( T_1, T_2, \ldots \) be the arrival times of \( N_s \), and suppose we observe these to be \( t_1, t_2, \ldots \), and \( N_t = n \). From class,
\[ P[T_1 \in (t_1, t_1 + dt), \ldots, T_n \in (t_n, t_n + dt)] = \lambda^n e^{-\lambda t} (dt)^n. \]
Thus, the value of \( \lambda \) at maximizes the likelihood must satisfy
\[ n\lambda^{n-1}e^{-\lambda t} - \lambda^n t e^{-\lambda t} = 0 \implies \lambda = \frac{n}{t}. \]

(b) \[ \text{Var}\left(\frac{N_t}{t}\right) = \frac{\lambda}{t^2}. \]

6. (a) Assume \( \lambda_1 > \lambda_0 \). By Problem 5, the likelihood-ratio is given by
\[ \Lambda((N_s, 0 \leq s \leq t)) = \frac{\lambda_1^{N_t} e^{-\lambda_1 t}}{\lambda_0^{N_t} e^{-\lambda_0 t}}, \]
so the Neyman-Pearson test is
\[ \hat{X} = \begin{cases} 1, \Lambda((N_s, 0 \leq s \leq t)) > \kappa \\ 1\{U > \gamma\}, \Lambda((N_s, 0 \leq s \leq t)) = \kappa \\ 0, \Lambda((N_s, 0 \leq s \leq t)) < \kappa \end{cases}, \]
where \( U \) is an independent r.v. uniform in \([0, 1]\), and \( \kappa, \gamma \) are determined by \( P[\hat{X} = 0|X = 1] \leq \beta \).

(b) We will use a suboptimal test to bound the errors. The test will be:
\[ \tilde{X} = \begin{cases} 1, \frac{N_t}{t} > \frac{\lambda_1 + \lambda_0}{2} \\ 0, \text{ otherwise} \end{cases} \]
Notice that for any \( a > 0 \),
\[ 1\{|N_t/t - \lambda_0| \geq a\} \leq \frac{(N_t/t - \lambda_0)^2}{a^2} 1\{|N_t/t - \lambda_0| \geq a\}, \]

We must show that these are mutually independent.

By Problem 5(b). Using \( a = (\lambda_1 - \lambda_0)/2 \), we can compute the error as

\[
P[\hat{X} = 1 | X = 0] \leq \frac{4\lambda_0}{t(\lambda_1 - \lambda_0)^2}.
\]

In the same way, we can compute

\[
P[\hat{X} = 0 | X = 1] \leq \frac{4\lambda_1}{t(\lambda_1 - \lambda_0)^2}.
\]

Thus, for \( t \geq 4 \max\{\lambda_1/\beta, \lambda_0/\alpha\}/(\lambda_1 - \lambda_0)^2 \), we have \( \max\{P[\hat{X} = 0 | X = 1], P[\hat{X} = 1 | X = 0]\} \leq \min\{\beta, \alpha\} \).

Since \( P[\hat{X} = 0 | X = 1] = \beta \geq P[\hat{X} = 0 | X = 1] \), the Neyman-Pearson test guarantees that

\[
P[\hat{X} = 1 | X = 0] \leq \frac{4\lambda_0}{t(\lambda_1 - \lambda_0)^2}.
\]

We may assume \( s \geq 0 \), since the setup is symmetric around 0.

Now, \( P(s - T_{n_0} > t) = E(P[s - T_{n_0} > t | n_0]) \). On \( n_0 = 0 \), \( P(s - T_{n_0} > t | n_0) = P(-T_0 > t - s) = 1(t > s) \). On \( n_0 = m > 0 \), the arrival times \((T_1, \cdots, T_m)\) are distributed as the order statistics of i.i.d. uniform on \([0, t]\), since \( N_+^+ = n_0 \).

Thus,

\[
P[s - T_{n_0} > t | n_0 = 0] = 1(t > s) \left(\frac{s - t}{s}\right)^m.
\]

Combining all the above yields,

\[
P(s - T_{n_0} > t) = P(N_s^+ = 0)\left(e^{-\lambda(t-s)}1\{t > s\} + 1\{t \leq s\}\right)
\]

\[
\quad + 1\{t < s\} \sum_{m=1}^{\infty} P(N_s^+ = m)\left(\frac{s - t}{s}\right)^m
\]

\[
= e^{-\lambda t}\left(e^{-\lambda(t-s)}1\{t > s\} + 1\{t \leq s\}\right)
\]

\[
\quad + 1\{t < s\} \sum_{m=1}^{\infty} \frac{e^{-\lambda s}(\lambda s)^m}{m!}\left(\frac{s - t}{s}\right)^m
\]

\[
= e^{-\lambda t}.
\]

We still need to show that \( T_{n+n_0} - T_{n+n_0-1} \) are i.i.d. \( \text{Exp}(\lambda) \), for all \( n \leq 0 \).

On \( T_{n+n_0} < 0 \), \( T_{n+n_0} - T_{n+n_0-1} \) is independent of \( (T_m : m \geq n + n_0) \) and exponentially distributed with rate \( \lambda \). This is because \( n_0 \) depends only on \( (T_m : m \geq 0) \), and \((T_m : m \leq 0)\) gives the arrivals of a Poisson process of rate \( \lambda \), which are independent of \( (T_m : m \geq 0) \).

Let’s assume \( T_{n+n_0} \geq 0 \) (so \( n + n_0 \geq 1 \)) now. Then,

\[
P[T_{n+n_0} - T_{n+n_0-1} > t | T_{n+n_0}] = E[P[T_{n+n_0} - T_{n+n_0-1} > t | T_{n+n_0}, n_0] | T_{n+n_0}]
\]

but \( \{T_{n+n_0} = u, N_u^+ = m, n_0 = k\} = \{T_{n+n_0} = u, n_0 = -n + m\} \), so

\[
P[T_{n+n_0} - T_{n+n_0-1} > t | T_{n+n_0} = u, n_0 = k, (N_r^+, u \leq r \leq s)]
\]

\[
P[u - T_{k+n} > t | T_{k+n} = u, N_u^+ = k + n, N_s - N_u^+ = -n, (N_r^+, u \leq r \leq s)]
\]

\[
P[u - T_{k+n} > t | N_u^+ = k + n - 1]
\]

\[
P[u - T_{n_0} > t | N_u^+ = k + n - 1],
\]

So taking expectations, conditional on \( X = 0 \), on both sides yields

\[
P[|N_t/t - \lambda_0| \geq a | X = 0] \leq \frac{E[(N_t/t - \lambda_0)^2 | X = 0]}{a^2}
\]

\[
= \frac{\lambda_0}{t a^2},
\]

by Problem 5(b). Using \( a = (\lambda_1 - \lambda_0)/2 \), we can compute the error as

\[
P[\hat{X} = 1 | X = 0] \leq \frac{4\lambda_0}{t(\lambda_1 - \lambda_0)^2}.
\]

In the same way, we can compute

\[
P[\hat{X} = 0 | X = 1] \leq \frac{4\lambda_1}{t(\lambda_1 - \lambda_0)^2}.
\]
where $N_{u-}^+ = \lim_{\epsilon \to 0} N_{u-}^+$. Thus, by letting $s = u$ in (1), $P[T_{n+n_0} - T_{n+n_0-1} > t | T_{n+n_0}] = e^{-\lambda t}$, where we have always conditioned on $\{T_n + n_0 \geq 0\}$.

Now, by combining the above,

$$P(T_{n+n_0} - T_{n+n_0-1} > t) = e^{-\lambda t}.$$ 

Also, (2) implies that $T_{n+n_0} - T_{n+n_0-1}$ is independent of $s - T_{n_0} (T_m - T_{m-1}, m = n + n_0 + 1, \ldots, n_0$).

Finally, note that $T_{n_0+1} - s, (T_{n+n_0+1} - T_{n+n_0}, n > 0)$ are i.i.d. Exp($\lambda$), independent of everything that came before time $s$. This follows from the fact that $(N_t^+ - N_s^+, t \geq 0)$ is a Poisson process, independent of $(N_r^+: 0 \leq r \leq s)$.

$E(T_1 - T_0) > E(T_{n+1} - T_n), \forall n \neq 0$ is called the “inspection” paradox: When an observer suddenly looks at a system of arrivals, he sees that the spread between the next and previous arrival, is “larger” than the average spread of the other inter-arrival times! The reason that this occurs is not particular to Poisson; it is because the observer is more likely to fell into a longer-than-usual interval. E.g., consider two types of alternating interarrival intervals; one that lasts 1 second, and another lasting 100 seconds. Although half of the intervals are short, there is a higher likelihood of encountering a long one because they occupy more time. In this example, the observed average interarrival interval is not $\frac{1}{2} 1s + \frac{1}{2} 100s = 50.5s$, but $\frac{1}{101} 1s + \frac{100}{101} 100s \approx 99s.$