Solutions to Problem Set 9

1. Assume the state-space is \{1, \ldots, N\}, and let \((V_n)\) be i.i.d. uniform in \([0, 1)\).
   For each state \(i\), define
   \[
   f(i, u) = \sum_{j=1}^{N} j \mathbf{1}\left\{ u \in \left[ \sum_{k=1}^{j-1} P(i, k), \sum_{k=1}^{j} P(i, k) \right) \right\},
   \]
   where \(\sum_{j=1}^{-1} j = 0\). Then,
   \[
P[X_{n+1} = j|X_n] = P\left[ V_n \in \left[ \sum_{k=1}^{j-1} P(i, k), \sum_{k=1}^{j} P(i, k) \right) \right] = P(i, j).
   \]

2. Since state-space is finite, jump rates are bounded so we can use uniformization to reduce the problem into a simulation of a discrete-time MC. Also, we will need a generator of i.i.d. exponential r.v’s. [C.f. Lecture 22, Section B.]

3. The balance equations are
   \[
   \begin{cases}
   \pi(0)(\lambda + \mu) = \pi(2)\lambda \\
   \pi(1)\mu = \pi(0)\lambda \\
   \pi(2)\lambda = \pi(1)\mu + \pi(0)\mu
   \end{cases}
   \]
   which together with \(\pi(0) + \pi(1) + \pi(2) = 1\), give
   \[
   \pi(0) = \frac{\mu \lambda}{(\lambda + \mu)^2}, \quad \pi(1) = \frac{\lambda^2}{(\lambda + \mu)^2}, \quad \text{and} \quad \pi(2) = \frac{\mu}{\lambda + \mu}.
   \]

4. Let \(N_t\) be a Poisson process with rate \(\lambda\), and \(Y_n\) an independent discrete-time MC with transition probabilities
   \[
P(i, j) = \begin{cases} 
   \frac{q(i, j)}{\lambda}, & i \neq j \\
   \frac{1}{1 + \frac{q(j, j)}{\lambda}}, & i = j.
   \end{cases}
   \]
Then for \( i \neq j \),

\[
P[Y_{t+\epsilon} = j|Y_t = i, (Y_{s \leq t}, s \leq t), N_t = n] = \\
P[Y_{n+1} = j, N_{t+\epsilon} - N_t = 1|Y_n = i, (Y_{m \leq n}, N_t = n] \\
+ \sum_{k=2}^{\infty} P[Y_{n+k} = j, N_{t+\epsilon} - N_t = k|Y_n = i, (Y_{m \leq n}, N_t = n]
\]

\[
= P[N_{t+\epsilon} - N_t = 1|Y_n = i, (Y_{m \leq n}, N_t = n] \\
\cdot P[Y_{n+1} = j|Y_n = i, (Y_{m \leq n}, N_t = n] \\
+ \sum_{k=2}^{\infty} P[N_{t+\epsilon} - N_t = k|Y_n = i, (Y_{m \leq n}, N_t = n] \\
\cdot P[Y_{n+k} = j|Y_n = i, (Y_{m \leq n}, N_t = n] \\
= P(N_{t+\epsilon})P[Y_{n+1} = j|Y_n = i] \\
+ \sum_{k=2}^{\infty} P(N_{t+\epsilon} - N_t = k)P[Y_{n+k} = j|Y_n = i]
\]

as \( \epsilon \downarrow 0 \). Similarly for \( i = j \).

5. (a) We give three different functions. The “time-change” representation,

\[
X_t = X_0 + N^\lambda(t) - N^\mu \left( \int_0^t 1\{X_s > 0\} ds \right).
\]

The “filtering” representation,

\[
X_t = X_0 + N^\lambda(t) - \int_0^t 1\{X_s > 0\} N^\mu(ds).
\]

The last integral is defined as \( \int_t^s g(N_{s-})N(ds) = \sum_{i=1}^{N_t} g(N_{T_{i}}) \), where \( T_1, T_2, \ldots \) are the arrival times of \( N_s \).

Finally, the “reflection mapping” representation

\[
X_t = N^\lambda(t) - N^\mu(t) + X_0 + \sup_{s \leq t} (N^\mu(s^{-}) - N^\lambda(s^{-}) - X_0^+).
\]

(Recall, \((x)^+ = \max(x, 0)\).)

Notice that the first two representations are given as differential equations, and the last two have the same sample-paths. We will consider the “filtering” representation. First, note that the equation defines a unique solution. E.g., to construct a solution, assuming that \( X_0 > 0 \), define \( X_t = X_0 + N^\lambda(t) - N^\mu(t) \) for \( t \in [0, T_1] \), where \( T_1 = \inf \{ t \geq 0 : N^\lambda(t) - N^\mu(t) = 0 \} \). This is the unique solution in \([0, T_1)\). Let \( T_2 = \inf \{ t > T_1 : N^\lambda(t) - N^\lambda(T_1) > 0 \} \), then \( X_t = 0 \) for \( t \in [T_1, T_2) \) is the unique solution in \([T_1, T_2)\). Hence, \((X_t, t \leq T_2)\) is the unique solution until time \( T_2 \). Continuing in this way, we show existence and uniqueness over \([0, \infty)\).

We now show that \( X_t \) given as above, has the desired distribution. Say \( X_t = 0 \), then

\[
P[X_{t+\epsilon} = 1|X_t = 0, N^\lambda(s), N^\mu(s) \leq t] = \\
P[N^\lambda(t + \epsilon) - N^\lambda(t) = 1, N^\mu(t + \epsilon) - N^\mu(t) = 0, X_t = 0, N^\lambda(s), N^\mu(s) \leq t] + o(\epsilon)
\]

\[
= \lambda \epsilon + o(\epsilon),
\]
In the general case, we can associate an independent Poisson process \( N_i^j(t) \) with rate 
\( q(i,j) \), to each transition from state \( i \) to state \( j \) (with \( q(i,j) > 0 \)). When the process is at 
\( X_t = 0 \) we “filter” out arrivals from all \( N_i^j(\cdot) \) with \( i \neq j \), and jump to state \( j \) on arrivals 
of \( N_i^j(\cdot) \). Then, we filter out all \( N_i^j(\cdot) \) with \( i \neq j \) and continue in the same way.

The rest of the transition matrix can be calculated in the same way.

(b) In the general case, we can associate an independent Poisson process \( N_{i,j}(t) \) with rate 
\( q(i,j) \), to each transition from state \( i \) to state \( j \) (with \( q(i,j) > 0 \)). When the process is at 
\( X_t = 0 \) we “filter” out arrivals from all \( N_i^j(\cdot) \) with \( k \neq i \), and jump to state \( j \) on arrivals 
of \( N_i^j(\cdot) \). Then, we filter out all \( N_i^j(\cdot) \) with \( k \neq j \) and continue in the same way.

6. Assume \( \tau_A < \infty \) so that what we want to show makes sense. Now,
\[
P[X_{\tau_A+\epsilon} = i \mid X_{\tau_A} = i; X_s, s \leq \tau_A] = E[P[X_{\tau_A+\epsilon} = i \mid X_{\tau_A} = i; X_s, s \leq \tau_A] \mid X_{\tau_A} = i; X_s, s \leq \tau_A]
\]
\[= q(i,j)\epsilon + o(\epsilon).\]
(Notice that the \( o(\epsilon) \) inside the expectation, was nonrandom.)

7. Let \( X_t, Y_t \) be two independent CTMC, both having rate matrix \( Q = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \). \( Y_t \) is initialized with the invariant distribution \((\pi(0), \pi(1)) = (\mu/(\lambda + \mu), \lambda/(\lambda + \mu))\).
Let \( T = \inf\{t \geq 0 : X_t = Y_t\} \). Conditional on \( X_0 = Y_0, T = 0 \) with probability one. Also, conditional on \( X_0 \neq Y_0, T = \text{Exp}(\lambda + \mu) \). Thus, \( T < \infty \) a.s. Now define the process \( Z_t \) as follows:
\[
Z_t = \begin{cases} 
X_t, & t \leq T \\
Y_t, & t > T.
\end{cases}
\]
This gives a MC with the same rate matrix \( Q \). To see this, note that
\[
P[Z_{t+\epsilon} = 1 \mid Z_t = 0, (Z_s, s \leq t)]
\]
\[
= P[Z_{t+\epsilon} = 1 \mid Z_t = 0, (Z_s, s \leq t), T \leq t] P[T \leq t] Z_t = 0, (Z_s, s \leq t)]
\]
\[+ P[Z_{t+\epsilon} = 1 \mid Z_t = 0, (Z_s, s \leq t), T > t] P[T > t] Z_t = 0, (Z_s, s \leq t)].
\]
But,
\[
P[Z_{t+\epsilon} = 1 \mid Z_t = 0, (Z_s, s \leq t), T \leq t] = P[Y_{t+\epsilon} = 1 \mid Y_t = 0] = \mu \epsilon + o(\epsilon),
\]
as \( \epsilon \downarrow 0 \). Also,
\[
P[Z_{t+\epsilon} = 1 \mid Z_t = 0, (Z_s, s \leq t), T > t] = P[Z_{t+\epsilon} = 1 \mid X_t = 0, (X_s, s \leq t), T > t]
\]
\[= P[Z_{t+\epsilon} = 1, (Y_r; r \in (t, t+\epsilon)) \text{ has no jumps, (} X_r; r \in (t, t+\epsilon) \text{) has exactly one jump} \]
\[\mid X_t = 0, Y_t = 1, (Y_s, s \leq t), (X_s, s \leq t), T > t \] + \( o(\epsilon) \)
\[= P[(Y_r; r \in (t, t+\epsilon)) \text{ has no jumps, (} X_r; r \in (t, t+\epsilon) \text{) has exactly one jump} \mid X_t = 0, Y_t = 1] + \epsilon(\epsilon)
\]
\[= \mu \epsilon + o(\epsilon).\]
The rates for other transitions -or no transitions- are obtained in a similar manner.
Now by construction, \( P(Z_0 = 0) = P(Y_0 = 0) \), so \( P(Z_t = 0) = P(X_t = 0) \) for all \( t \geq 0 \). So,
\[
|P(X_t = 0) - \pi(0)| = |P(Z_t = 0) - P(Y_t = 0)|
\]
\[= |P(Z_t = 0, T > t) - P(Y_t = 0, T > t) + P(Z_t = 0, T \leq t) - P(Y_t = 0, T \leq t)| \leq P(T > t) \rightarrow 0,
\]
as \( t \rightarrow \infty \), since \( T < \infty \).