Sample quiz problems

Problem 1
Compute max\(\{3x^2 + 2\sqrt{2}xy + 4y^2 : x^2 + y^2 = 1, x, y \in \mathbb{R}\}\).

Notice that the expression to be maximized can be written differently as
\[
[x \ y] \begin{bmatrix}
3 & \sqrt{2} \\
\sqrt{2} & 4
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}.
\]

The eigenvalues of the above matrix (call it \(A\)) are the solutions of \((\lambda - 3)(\lambda - 4) - 2 = 0\). These are \(\lambda_1 = 5\) and \(\lambda_2 = 2\). Since \(A\) is positive definite, it has a decomposition \(A = V \Lambda V^*\), where \(V\) is unitary and \(\Lambda = \text{diag}(5, 2)\). So if we change coordinates by setting \([w \ z] = [x \ y] V\), then \((1)\) equals \(5w^2 + 2z^2\). But the unit circle \(\{(x, y) : x^2 + y^2 = 1\}\) is invariant under rotations, so \(x^2 + y^2 = w^2 + z^2 = 1\) for \((x, y)\) in this set. Therefore the maximum value of \(5w^2 + 2z^2\) is achieved for \((w, z) = (1, 0)\), and is 5.

Problem 2
Compute the singular value decomposition of the matrix
\[
A = \begin{bmatrix}
1 & 3 \\
4 & 0
\end{bmatrix}.
\]

Assume the existence of decomposition \(A = U \Sigma V^*\), where \(\Sigma\) is diagonal and \(U, V\) unitary. Then, \(A^*AV = V \Sigma U^*U \Sigma V^*V = V \Sigma^2\), so the columns of \(V = [v_1 \ | \ v_2]\) must be the normalized eigenvectors of
\[
A^*A = \begin{bmatrix}
17 & 3 \\
3 & 9
\end{bmatrix}.
\]

Now, \(\det(A^*A - \lambda I) = (17 - \lambda)(9 - \lambda) - 9\) which has two solutions, \(\sigma_1^2 = 18\) and \(\sigma_2^2 = 8\). Solving \(A^*Av_1 = 18v_1\), \(A^*Av_2 = 8v_2\) yields,
\[
v_1 = \frac{1}{\sqrt{10}} \begin{bmatrix}
3 \\
1
\end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{10}} \begin{bmatrix}
1 \\
-3
\end{bmatrix}.
\]
You could do the same for $U$, i.e., noticing $AA^*U = U\Sigma^2$ and proceed solving for $U$. But this is not necessary since $U$ is uniquely defined by $A = U\Sigma V^*$:

\[
U = AV\Sigma^{-1} = \begin{bmatrix}
1 & 3 \\
4 & 0
\end{bmatrix} \begin{bmatrix}
3/\sqrt{10} & 1/\sqrt{10} \\
1/\sqrt{10} & 1/\sqrt{8}
\end{bmatrix} \begin{bmatrix}
1/\sqrt{18} & 1/\sqrt{5} \\
2 & 1
\end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix}
1 & 2 \\
-2 & 1
\end{bmatrix}.
\]

**Problem 3**

Let $X, Y, Z$ be three random variables. Show that

\[
E[(X - E[X|Y])^2] \geq E[(X - E[X|Y,Z])^2].
\]

First solution: using the characterization of conditional expectation as a projection. According to this, $E[X|Y] = g^*(Y)$ where $g^*$ is the function that minimizes $E[(X - g(Y))^2]$ over all $g$ with $E[g(Y)^2] < \infty$. Similarly, $E[X|Y,Z] = h^*(Y,Z)$ is the projection on $\{h(Y,Z) : h\text{ any real function s.t. }E[h(Y,Z)^2] < \infty\}$. But $\{g(Y) : g\text{ any real function s.t. }E[g(Y)^2] < \infty\}$ is a linear subspace of $\{h(Y,Z) : h\text{ any real function s.t. }E[h(Y,Z)^2] < \infty\}$. Thus, $E[(X - E[X|Y])^2] \geq E[(X - E[X|Y,Z])^2]$, by the property of projection.

Second solution: if you don’t know the projection characterization above, you could start from “algebraic” properties of conditional expectation and essentially derive the Pythagorean theorem. This is why the projection characterization is always useful to have in mind.

Let $X_1 = E[X|Y]$ and $X_2 = E[X|Y,Z]$. Note that

\[
E[(X - X_2)(X_2 - X_1)] = E[E[(X - X_2)(X_2 - X_1)|Y,Z]],
\]

and

\[
E[(X - X_2)(X_2 - X_1)|Y,Z] = (X_2 - X_1)E[X - X_2|Y,Z] = X_2 - X_1 = 0.
\]

Hence,

\[
E[(X - X_1)^2] = E[(X - X_2 + X_2 - X_1)^2] = E[(X - X_2)^2] + E[(X_2 - X_1)^2] \geq E[(X - X_2)^2].
\]