EE228a - Lecture 17 - Spring 2006
Non-cooperative Games

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Abstract

I. Overview

The goal of this lecture is to learn a generalized game structure, the notion of mixed strategy, Nash equilibrium, and calculation of expected payoffs when the information is imperfect, with examples of basic and famous games: Matching pennies, Cournot duopoly, and Principal-Agent games. Specifically we will review

- One-shot (Static) game:
- Nash Equilibrium
- Cournot
- Leader-Follower (Stackelberg)
- Bayes-Cournot
- Principal-Agent Problem
- Existence of NE

II. Terminology

It will be beneficial to define some frequent terminology. Rest of it will be defined inline.

- A one-shot game is a game occurring only once (so players walk away and never meet again.) A static game is a synonym for a one-shot game. A example of a static game is the ‘Matching pennies’ game.
- All games can be classified as complete information games or incomplete information games.
- In complete information games, the player whose turn it is knows at least as much as those who moved before him. Complete information games include perfect information games and imperfect information games.
- In perfect information games, players know the full history of the game, all moves made by all players and all payoffs.
- In imperfect information games, players know all the possible outcomes/payoffs, but not the actions chosen by other players.
- In incomplete information games, the player whose turn it is knows less that a player who has already moved.
- The utility is a synonym of payoff.

III. Matching Pennies Game

Let’s recall this game.

Game statement 1: Matching pennies

Each of Alice and Bob has one penny. Simultaneously they show a penny to each other. If the faces match, Bob gives Alice $1.00. If the faces do not match, Alice gives Bob $1.00. This is an one-shot game with simultaneous move and complete information about the reward function.

We can generalize the actions of the players and the reward function of each player of the above simple game. Let {Left, Right} be the set of possible actions of Bob and let {Top, Bottom} be the set of possible actions of Alice. Let \( A(\text{Alice’s action, Bob’s action}) \) be a reward function for Alice, given both her choice of action and bob’s. Similarly, let \( B(\text{Alice’s action, Bob’s action}) \) be a reward function for Bob. Employing only the initials of actions, matching pennies game will be generalized as below.

Game statement 2: Mixed strategy of Matching pennies

A mixed strategy is one in which a player plays his available pure strategies with certain probabilities. The previous matching pennies game was an example of a pure strategy game; the player chooses one action with probability 1. Now we can extend this pure strategy game to a mixed strategy game by providing a probability distribution on the strategy of the player. Let \( s_j \) be a strategy of player \( j \). A set of feasible strategies of player \( j \) is denoted by \( S_j \) and \( s_j \in S_j \). We define the reward of player \( i \) as
a function of actions of all players: \( u_i(s_1, \cdots, s_N) \), where \( N \) is the number of the players. The vector \( s \) denotes \( (s_1, \cdots, s_N) \). The strategy of player \( i \) follows a probability distribution \( \pi_i \) on \( S_i \). Then we can say \( s \) follows a joint distribution of \( \pi_k \)'s. That is,

\[
s = D \pi_1 \times \cdots \times \pi_N
\]

In general, the goal of player \( i \) is defined as

\[
\text{Maximize } E(u_i(\text{textbfs})).
\]

**Example 1:** Let’s look at an example. Let \( a \) be the probability that Alice chooses the strategy \( H \) and let \( b \) be the probability that Bob chooses the strategy \( H \). \( 1 - a \) and \( 1 - b \) are defined similarly. Then the expected reward \( u_A \) for Alice can be calculated as

\[
u_A = ab + (1 - a)(1 - b) - a(1 - b) - (1 - a)b
\]

\[
= 1 - 2a - 2b + 4ab
\]

\[
= 1 - 2b + 2a(2b - 1)
\]

\[
a = \begin{cases} 
1, & b > 1/2 \\
0, & b < 1/2 \\
\text{any value,} & b = 1/2
\end{cases}
\]

In the randomized case, \( s = (s_1, \cdots, s_N) \) is a Nash equilibrium if \( \pi \) s.t. \( \pi_i \) maximizes \( E(u_i) \) over \( \pi_i \) for all \( i \).

**Definition 1:** Nash Equilibrium

A selfish (or rational as an euphemism) player will try to get the best payoff for himself. This best response of each player is affected by a particular strategy choice of the other players. If each player’s strategy choice is a best response to that of the other player, then we have found a solution or equilibrium to the game. This solution concept is known as a Nash equilibrium, after John Nash who proposed it first.

A game may have zero, one or multiple Nash equilibria. A Nash equilibrium has an essential descriptive feature: no player has an incentive to deviate unilaterally from that equilibrium. In the non-randomized case, \( s = (s_1, \cdots, s_N) \) is a Nash equilibrium if

\[
u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall i,
\]

where \( s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_N) \).

In the randomized case, \( s = (s_1, \cdots, s_N) \) is a Nash equilibrium if

\[
\pi \text{ s.t. } \pi_i \text{ maximizes } E(u_i) \text{ over } \pi_i \text{ for all } i.
\]

**IV. COURNOT DUOPOLY GAME**

Let’s explore a new kind of game called Cournot duopoly. We want to find the Nash equilibrium and maximal expectation of the reward of each player in the Cournot duopoly game. Let’s recap the Cournot game in breif.

**Game statement 3:** Cournot duopoly

Players are two firms. They produce quantity \( q_1 \) and \( q_2 \) of the same product (commodity) respectively. The unit price of the product in market is determined as \( A - q_1 - q_2 \), where \( A \) is a positive constant. For \( i = 1, 2 \), the profit of the firm \( i \) is defined as

\[
u_i = q_i(A - q_1 - q_2) - Cq_i.
\]

where \( C \) is the unit production cost.
**A. Nash equilibrium of Cournot duopoly**

Now, let’s define $B = A - C$ and calculate $u_2$.

$$u_2 = q_2(A - q_1 - q_2 - C) \quad (11)$$

To maximize $u_2$ we differentiate both sides and set the derivative to zero.

$$\frac{du_2}{dq_2} = (A - q_1 - C) - 2q_2 = 0 \quad (12)$$
$$q_2 = (A - q_1 - C)/2 \quad (13)$$
$$= (B - q_1)/2 \quad (14)$$

Similarly, $q_1 = (B - q_2)/2 \quad (15)$

Therefore, we have $q_1^* = B/3 = q_2^* \quad (16)$

$$u_i^* = q_i^*(B - q_i^*) = B^2/9, \text{ for all } i. \quad (17)$$

![Best response functions](image)

Fig. 3. Best response function and Nash equilibrium of Cournot duopoly

The best response function of each firm the equilibrium is illustrated in Fig.[3].

**B. Cooperation Cournot**

**Game statement 4: Cooperation Cournot**

In this game the firms cooperate for their best payoffs and split the revenues.

The payoff of each firm is defined as before. Then they choose $q = q/2$ where $q$ maximizes

$$u = u_1 + u_2 = 2u_i = q(B - q) \quad (18)$$

To maximize $u$ we differentiate both sides and set the derivative to zero.

$$\frac{du}{dq} = B - 2q = 0 \quad (19)$$
$$q = B/2 \quad (20)$$
$$q_i^* = B/4 \quad (21)$$

$$u_i^* = B^2/8 \text{ for all } i. \quad (22)$$

Comparing these results with the results under the Nash equilibrium (where $q_i = B/3$ and $u_i = B^2/9$), the firms under Cooperation Cournot produce less and make more profit than they are under pure competition. (Note: less production implies less cost.) This gives us an intuitive clue why corporations sometimes try to merge with others.

**C. Stackelberg Cournot**

There are many situations where a firm takes the lead and others follow. Such leader-follower games are called Stackelberg games.

**Game statement 5: Stackelberg Cournot**

One player announces its strategy and others follow.

Assume firm 1 announces $q_1$ and firm follows. Then the firm 2 will try to maximize its own payoff based on this given information.

$$u_2 = q_2(B - q_1 - q_2) \quad (24)$$

$$\frac{du_2}{dq_2} = (B - q_1) - 2q_2 = 0 \quad (25)$$

$$q_2 = (B - q_1)/2 \quad (26)$$

So the firm 2 determines its production quantity as a function of $q_1$. The firm 1 knows that the firm 2 will choose its $q_2$ as above. Given this rationale, the firm 1 will try to maximize its own payoff.
Thus the payoff of the firm 2 in terms of production cost. However, we can calculate the expected maximum payoff instead. Let’s find the Nash equilibrium. As before,

\[ u_2 = q_2(A - q_1 - q_2 - C_2) \]  
\[ \frac{du_2}{dq_2} = (A - q_1 - C_2) - 2q_2 = 0 \]  
\[ q_2 = (A - q_1 - C_2)/2 \]

Similarly, \( q_1 = (A - q_2 - C_1)/2 \)

This gives us another intuition: under the Stackelberg of Cournot’s duopoly, the leader has the advantage.

D. Imperfect information Cournot games

Now assume that the production cost is each firm’s secret and not shared with others. A firm knows its own production cost deterministically but knows other firm’s production cost only as a probabilistic distribution. We call this Bayesian Cournot game.

Firm 1 knows \( C = E(C_2), \sigma^2 = \text{var}(C_2) \)
Firm 2 knows \( C = E(C_1), \sigma^2 = \text{var}(C_1) \)

Let’s find the Nash equilibrium. As before,

\[ u_2 = q_2(A - q_1 - q_2 - C_2) \]  
\[ \frac{du_2}{dq_2} = (A - q_1 - C_2) - 2q_2 = 0 \]  
\[ q_2 = (A - q_1 - C_2)/2 \]

Similarly, \( q_1 = (A - q_2 - C_1)/2 \)

Because of the imperfect information on the production cost, no firm is able to solve the Nash equilibrium to get the maximum payoff. However, we can calculate the expected maximum payoff instead. The expected payoff for the firm 2 is defined as \( E[u_2|q_2, C_2] \). Assume the firm 2 tries to maximize it.

\[ E[u_2|q_2, C_2] = q_2(A - Q_1 - q_2 - C_2), \text{ where } Q_1 = E(q_1). \]  
\[ \frac{dE[u_2|q_2, C_2]}{dq_2} = (A - Q_1 - C_2) - 2q_2 = 0 \]  
\[ q_2 = (A - Q_1 - C_2)/2 \]

Taking the expectation of both sides yields

\[ Q_2 = (A - Q_1 - C)/2 = (B - Q_1)/2 \]

where \( B = A - C \)

Similarly \( Q_1 = (B - Q_2)/2 \)

Therefore \( Q_1 = Q_2 = Q = B/3 \)

This gives \( q_2 = (A - B/3 - C_2)/2 \)
\( q_1 = (A - B/3 - C_1)/2 \)

Therefore the payoff of the firm 2 in terms \( C_1 \) and \( C_2 \) is

\[ u_2 = (A - B/3 - C_2)(B/3 + C_1/2 - C_2/2)/2 \]

Thus

\[ E(u_2) = E(E(u_2|C_2)) \]
\[ = E((A - B/3)B/3 - C_2B/3 + (A - B/3)C/2 - CC_2/2 - (A - B/3)C_2/2 - C_2^2/2)/2 \]
\[ = B^2/9 + \sigma^2/4 \]
E. Comparison of Nash equilibrium: Complete(full) information case

Let’s assume costs $C_i$ follow a certain distribution with the first and second order statistics as before and independent. But now the both costs are known to all firms.

$$u_2 = q_2(A - q_1 - q_2 - C_2)$$  \hspace{1cm} (56)

Differentiating both sides and setting the derivative to zero gives

$$\frac{du_2}{dq_2} = (A - q_1 - C_2) - 2q_2 = 0$$  \hspace{1cm} (57)

$$q_2 = (A - q_1 - C_2)/2$$  \hspace{1cm} (58)

Similarly, $q_1 = (A - q_2 - C_1)/2$  \hspace{1cm} (59)

Therefore, $q_1 = (A + C_2 - 2C_1)/3$  \hspace{1cm} (60)

and $q_2 = (A + C_1 - 2C_2)/3$  \hspace{1cm} (61)

Hence $u_1 = (A + C_2 - 2C_1)^2/9$  \hspace{1cm} (62)

and $u_2 = (A + C_1 - 2C_2)^2/9$  \hspace{1cm} (63)

The expected payoff for firm 2 becomes

$$E(u_2) = E(E(u_2|C_2))$$  \hspace{1cm} (64)

$$= B^2/9 + 5\sigma^2/9 \text{ where } B = A - C.$$  \hspace{1cm} (65)

If we recall the expected payoff in the incomplete information case, it was $E(u_2) = B^2/9 + 5\sigma^2/9$. Thus, the price of lack of information is regarded as $11\sigma^2/36$.

F. Imperfect information Stackberg Cournot game

We can repeat the same analysis for Stackberg Cournot game as done for Nash equilibrium of Bayesian Cournot game.

$$u_1 = q_1(A - q_1 - q_2 - C_1)$$  \hspace{1cm} (66)

$$u_2 = q_2(A - q_1 - q_2 - C_2)$$  \hspace{1cm} (67)

Now assume the firm 1 announces its production quantity $q_1$.

$$\frac{du_2}{dq_2} = (A - q_1 - C_2) - 2q_2 = 0$$  \hspace{1cm} (68)

$$q_2 = (A - q_1 - C_2)/2$$  \hspace{1cm} (69)

Hence, $u_1 = q_1(A - q_1 + C_2 - 2C_1)/2$  \hspace{1cm} (70)

The expected payoff for firm 1 becomes

$$E[u_1|q_1, C_1] = q_1(A - q_1 + C_2 - 2C_1)/2$$  \hspace{1cm} (71)

To maximize, $dE[u_1|q_1, C_1] / dq_1 = (A + C - 2C_1) - 2q_1 = 0$  \hspace{1cm} (72)

$$q_1 = (A + C - 2C_1)/2$$  \hspace{1cm} (73)

Hence $u_1 = (A + C - 2C_1)(A - (A + C - 2C_1)/2 + C_2 - 2C_1)/4$  \hspace{1cm} (74)

$$= (B + 2(C - C_1))(B + 2(C_2 - C_1))/8$$  \hspace{1cm} (75)

The expected payoff for firm 2 becomes

$$E(u_1) = E(E(u_1|C_1))$$  \hspace{1cm} (76)

$$= E(E(B^2 + 2B(C - C_1) + 2B(C_2 - C_1) + 4C^2 - 8CC_1 + 4C_1^2|C_1))/8$$  \hspace{1cm} (77)

$$= E(B^2 + 4B(C - C_1) + 4C^2 - 8CC_1 + 4C_1^2)/8$$  \hspace{1cm} (78)

$$= B^2/8 + \sigma^2/2$$  \hspace{1cm} (79)

Similarly, $E(u_2) = B^2/16 + \sigma^2/2$  \hspace{1cm} (80)

For the full information sharing case, we can repeat a very similar calculation and get

$$E(u_1) = B^2/8 + 5\sigma^2/8$$  \hspace{1cm} (81)

$$E(u_2) = B^2/16 + 13\sigma^2/16$$  \hspace{1cm} (82)

The cost of lack of information is $\sigma^2/8$ to the firm 1 and $5\sigma^2/16$ to the firm 2.

G. Summary of Cournot

The payoffs are summarized in the table [II].
TABLE II
PAYOFF COMPARISON

<table>
<thead>
<tr>
<th></th>
<th>Full</th>
<th>Bayes</th>
<th>Δ</th>
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</thead>
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<tr>
<td>LF u₁</td>
<td>(B^2/8 + 5\sigma^2/8)</td>
<td>(B^2/8 + \sigma^2/8)</td>
<td>(\sigma^2/8)</td>
</tr>
<tr>
<td>LF u₂</td>
<td>(B^2/16 + 13\sigma^2/16)</td>
<td>(B^2/16 + \sigma^2/2)</td>
<td>(5\sigma^2/16)</td>
</tr>
<tr>
<td>Nash</td>
<td>(B^2/9 + 5\sigma^2/9)</td>
<td>(B^2/9 + \sigma^2/4)</td>
<td>(11\sigma^2/36)</td>
</tr>
<tr>
<td>Coop</td>
<td>(B^2/8)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

V. PRINCIPAL-AGENT GAME

Game statement 6: Principal-Agent The principal-agent game is a class of games of imperfect information in which one player (the principal) attempts to offer incentives to the other (the agent) to encourage the agent to act in the principal’s best interest.

The goal of the game is to maximize the utility of principal. The principal offers the incentive \(t\) to the agent. The agent can provide \(q\) products whose marginal production cost per unit, named agent’s type, is \(\theta \in \Theta\) which is the agent’s type space. The principal makes use of \(q\) products provided from the agent and gets the utility \(S(q) - t\). For a plausible model, we require \(S'(q) > 0, S''(q) < 0, S(0) = 0\). Meanwhile the agent’s utility is given as \(t - F - \theta q\), where \(F\) is a fixed production cost. The type \(\theta\) is the agent’s private information and unobservable to the principal. It is declared by the agent during the contract. The incentive offered by principal is a function of \(\theta\). It is assumed that the incentive space and the type space are compact spaces. The production quantity \(q\) also is a function of \(\theta\).

When \(\theta\) is given, principal can calculate the socially efficient production \(q_\text{s}(\theta)\) to maximize the social well fare,

\[
\text{Max}\{\text{Principal’s utility} + \text{Agent’s utility}\} = S(q) - F - \theta q
\]

\[
\rightarrow S'(q_\text{s}(\theta)) = 0
\]

This is only feasible only if

\[
S(q_\text{s}(\theta)) > F + \theta q_\text{s}(\theta)
\]

A. Contract

We define the contract as the pair of incentive that principal offers to the agent and the corresponding production quantity that the agent provides back to principal: \((t(q), q_\text{s}(\theta))\). From now on we assume \(F = 0\).

Agents can be of two types: efficient \(\theta_L\), or inefficient \(\theta_H > \theta_L\). Since the type is declared by the agent and principal is incapable of verity this claim, the contract should be well designed to avoid moral hazards. In other words, we need design contract \((t_L, q_L), (t_H, q_H)\) so that the agent weakly prefer \((t_L, q_L)\) if his type is \(\theta_L\).

The contract is said to be incentive compatible if the agent’s utility is greater when the agent does not cheat on the type. That is,

\[
t_L - \theta_L q_L > t_H - \theta_L q_H
\]

and

\[
t_H - \theta_H q_H > t_L - \theta_H q_L
\]

The contract is said to be feasible if the agent’s utility is non-negative. That is,

\[
U_k := t_k - \theta_k q_k \geq 0, \text{ for } k \in H, L
\]

B. Bayesian optimal contract

Assume principal exactly does not know that the agent is efficient or inefficient. Let \(\nu := P(\theta = \theta_L)\) and \(1 - \nu := P(\theta = \theta_H)\). Now we want to find one key factor of the contract, \(q_\nu(\theta)\) such that it maximizes the principal’s expected utility. Then

\[
\text{Maximize } E(S(q) - t) = E(S(q) - \theta q - U)
\]

\[
= E(\text{Social value}) - E(\text{Rent of Agent})
\]

\[
q_\nu(\theta_L) = q_\text{s}(\theta_L)
\]

\[
q_\nu(\theta_H) < q_\text{s}(\theta_H)
\]

\[
S'(q_\nu(\theta_H)) = \theta_L + \frac{\nu}{1 - \nu}(\theta_H - \theta_L)
\]

Note that

\[
U_H = 0
\]

\[
U_L = (\theta_H - \theta_L)q_\nu(\theta_H)
\]

\(U_L\) is the rent that the efficient agent can get by mimicking the inefficient agent. Thus, there is a tradeoff between efficiency and information rent.
C. Revelation principle

The moral of the revelation principle is as follows: as contracting with the principal, the agent may be tempted to try to induce more utility by revealing the function of the type, \( m(\theta) \), instead of directly revealing the type, \( \theta \). According to the revelation principle, indirect revealing does not go one better than direct revealing. This is the justification, whereby literature on mechanism design usually concentrates upon the direct mechanism.

Proof: Let a direct contract be \( (t(\theta), q(\theta)) \) and an indirect contract be \( (T(m(\theta)), Q(m(\theta))) \) under the same principal-agent game. Pick any \( m, T, Q \). Then define composite functions \( t = T \circ m \) and \( q = Q \circ m \).

D. Existence of Nash equilibrium

Theorem 1: Existence of Nash equilibrium

Every finite static game has at least one Nash equilibrium (possibly randomized).

Proof: Let \( f(\pi) \) be the set of best responses to \( \pi \). Note that \( f(\pi) \) is convex by linearity of expectation. Also, \( f(\cdot) \) is graph continuous:

If \( s_n \in f(u_n) \) and \((u_n, s_n) \to (u, s)\), then \( s \in f(u) \) by continuity.

Conclusion follows from Kakutani’s fixed point theorem:

There must be some \( u \) such that \( u \in f(u) \).