Cooperative Game Theory

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What is Desirable?

• We’ve seen that
  – Prisoner’s Dilemma has undesirable Nash Equilibrium.
  – One shot Cournot has a less than socially optimum equilibrium.

• In a repeated game with threat strategies
  – Players can reach a more desirable equilibrium.

• We now classify different types of “Desirable” Equilibria.
Feasible Payoffs

Consider a two player game with a Feasible region of Payoffs:
Dominated Points and Pareto Efficiency

Feasible payoffs

- Vectors for which one player’s rewards can be increased without decreasing the others are dominated.

- Vectors which are not dominated, are Pareto Efficient.
A Social Optimum Vector Maximizes the Sum of Player Payoffs.
A max-min fair share vector is such that one player’s reward cannot be increased without decreasing the reward to another who already has less.
Nash Bargaining Equilibrium

- A Feasible Allocation Satisfying

\[ \sum_j \frac{x_j - x_j^N}{x_j^N} \leq 0. \]

- This equilibrium maximizes the Product of Player Payoffs.

- Soon, we will look at Bargaining Problem that this solves...
Example of Different Pareto Efficient Solutions:

- Social Optimum: (0,1/2,1/2)
- Max-Min: (1/2,1/2,1/2)
- NBE: (1/3,2/3,2/3)
Nash’s Bargaining Problem

• Model
  - Two players with interdependent payoffs $U$ and $V$
  - Acting together they can achieve a set of feasible payoffs
  - The more one player gets, the less the other is able to get
  - And there are multiple Pareto efficient payoffs

• Q: which feasible payoff would they settle on?
  - Fairness issue

• Example (from Owen):
  - Two men try to decide how to split $100$
  - One is very rich, so that $U(x) \equiv x$
  - The other has only $1$, so $V(x) \equiv \log(1+x) - \log 1 = \log(1+x)$
  - How would they split the money?
Intuition

• Feasible set of payoffs
  - Denote $x$ the amount that the rich man gets
  - $(u,v) = (x, \log(101 - x)), \ x \in [0,100]$

A fair split should satisfy

\[ \left| \frac{\Delta u}{u} \right| = \left| \frac{\Delta v}{v} \right| \]

Let $\Delta \to 0$, $\frac{du}{u} = -\frac{dv}{v}$
Or $\frac{du}{u} + \frac{dv}{v} = 0$, or
\[ vd\text{d}u + ud\text{d}v = 0, \text{ or } d(uv) = 0. \]

$\Rightarrow$ Find the allocation which maximizes $U \times V$

$\Rightarrow x^* = 76.8!$
Nash’s Axiomatic Approach (1950)

- A solution \((u^*, v^*)\) should be
  - Rational
    - \((u^*, v^*) \geq (u_0, v_0)\), where \((u_0, v_0)\) is the worst payoffs that the players can get.
  - Feasible
    - \((u^*, v^*) \in S\), the set of feasible payoffs.
  - Pareto efficient
  - Symmetric
    - If \(S\) is such that \((u, v) \in S \iff (v, u) \in S\), then \(u^* = v^*\).
  - Independent from linear transformations
  - Independent from irrelevant alternatives
    - Suppose \(T \subseteq S\). If \((u^*, v^*) \in T\) is a solution to \(S\), then \((u^*, v^*)\) should also be a solution to \(T\).
Results

• There is a unique solution which
  - satisfies the above axioms
  - maximizes the product of two players’ additional payoffs
    \[(u-u_0)(v-v_0)\]

• This solution can be enforced by “threats”
  - Each player independently announces his/her threat
  - Players then bargain on their threats
  - If they reach an agreement, that agreement takes effect
  - Otherwise, initially announced threats will be used

• Different fairness criteria can be achieved by changing the last axiom (see references)
Nash Bargaining Equilibrium

Finally, \( x^N \) is a Nash Bargaining Equilibrium if for all \( x \in R \) one has

\[
\sum_j \frac{x_j - x_j^N}{x_j^N} \leq 0.
\]

- Maximizes Product of

\[
\prod_j x_j = \prod_j (x_j^N + (x_j - x_j^N))
\]

\[
= \prod_j x_j^N \times \prod_j \left(1 + \frac{x_j - x_j^N}{x_j^N}\right)
\]

\[
= \prod_j x_j^N \times \left(1 + \sum_j \frac{x_j - x_j^N}{x_j^N} + \ldots\right)
\]
Nash Bargaining Equilibrium

Feasible payoffs
Suggested Readings

  – Nash’s original paper. Very well written.
  – A paper which unifies all bargaining solutions into a single framework
  – Applies Nash’s bargaining solution to resource allocation problem in admission control (multi-objective optimization)
Coalitions

• Model
  - Players \( n > 2 \) form coalitions among themselves
  - A coalition is any nonempty subset of \( N \)
  - Characteristic function \( V \) defines a game
    \[
    V(S) = \text{payoff to } S \text{ in the game between } S \text{ and } N - S, \ \forall S \subset N
    \]
    \[
    V(N) = \text{total payoff achieved by all players acting together}
    \]
    \[
    V(\cdot) \text{ is assumed to be super-additive}
    \]
    \[
    \forall S, T \subset N, \ V(S+T) \geq V(S)+V(T)
    \]

• Questions of Interest
  - Condition for forming stable coalitions
  - When will a single coalition be formed?
    - How to distribute payoffs among players in a fair way?
Core Sets

• Allocation \( X=(x_1, \ldots, x_n) \)
  \[ x_i \geq V(\{i\}), \quad \forall \ i \in N; \quad \sum_{i \in N} x_i = V(N). \]

• The core of a game
  
  any allocation which satisfies \( \sum_{i \in S} x_i \geq V(S), \quad \forall S \subseteq N \)

  \[ \Rightarrow \text{If the core is nonempty, a single coalition can be formed} \]

• An example
  
  • A Berkeley landlord (L) is renting out a room
  • Al (A) and Bob (B) are willing to rent the room at $600 and $800, respectively
  • Who should get the room at what rent?
Example: Core Set

- Characteristic function of the game
  - These combos give no payoff:
    \[ V(L) = V(A) = V(B) = V(A+B) = 0 \]
  - Coalition between \( L \) and \( A \) or \( L \) and \( B \)
    
    If rent = \( x \), then \( L \)'s payoff = \( x \), \( A \)'s payoff = \( 600 - x \)
    
    so \( V(L+A) = 600 \). Similarly, \( V(L+B) = 800 \)
  - Coalition among \( L \), \( A \) and \( B \): \( V(L+A+B) = 800 \)

- The core of the game:
  \[
  \begin{align*}
  x_L + x_A & \geq 600 \\
  x_L + x_B & \geq 800 \\
  x_L + x_A + x_B & = 800
  \end{align*}
  \]
  \[
  \Rightarrow \quad \text{core} = \{ (x_L = y, x_A = 0, x_B = 800 - y), \quad 600 \leq y \leq 800 \}
  \]

- B should get the place, and the rent should be between $600 and $800
Shapley Value: Example

• Consider
  - Landowner
  - 2 Farm Workers

• A Landowner + One Worker \(\rightarrow C\)
• A Landowner + Two Workers \(\rightarrow 2C\)
• One or Two Workers + No Landowner \(\rightarrow 0\)

• How much should each get?
  - We argue C for the landowner and C/2 for each worker.
Shapley Value: Example

- Imagine the parties arrive in random order, and each gets their marginal contribution.
  - ORDER             MARGINAL CONTRIBUTION
  - (F,W,W) → (0,C, C)
  - (W,F,W) → (0,C, C)
  - (W,W,F) → (0,0,2C)

- Farmer Avg = \( \frac{1}{6}( 2 \times 0 + 2 \times C + 2 \times 2XC) = C \)
- Worker Avg = \( \frac{1}{6}( 2 \times C + C + 0) = \frac{C}{2} \)
Fair Allocation: the Shapley Value

- Define solution for player $i$ in game $V$ by $P_i(V)$
- Shapley’s axioms
  - $P_i$’s are independent from permutation of labels
  - Additive: if $U$ and $V$ are any two games, then
    \[ P_i(U+V) = P_i(U) + P_i(V), \forall i \in N \]
  - $T$ is a carrier of $N$ if $V(S \cap T) = V(S)$, $\forall S \subseteq N$. Then for any carrier $T$, $\sum_{i \in T} P_i = V(T)$.
- Unique solution: Shapley’s value (1953)
  \[ P_i = \sum_{S \subseteq N} \frac{(|S|-1)! \cdot (N-|S|)!}{N!} [V(S) - V(S - \{i\})] \]
- Intuition: a probabilistic interpretation
Suggested Readings

  – Shapley’s original paper.

  – Applies Shapley’s value to caller -ID service.

  – How coalition could improve the revenue of international telephone carriers.
Summary

• Models
  - Strategic games
    ▪ Static games, multi-stage games
  - Cooperative games
    ▪ Bargaining problem, coalitions

• Solution concepts
  - Strategic games
    ▪ Nash equilibrium, Subgame-perfect Nash equilibrium
  - Cooperative games
    ▪ Nash’s solution, Shapley value

• Application to networking research
  - Modeling and design
References

  – an easy-to-read introductory to the subject
  – a concise but rigorous treatment on the subject
  – a good reference on cooperative games
  – a complete handbook; “the bible for game theory”