Stochastic Differential Equations for Power Law Behaviors

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Exponential Growth with Exponential Restart

\[ dX_t = \mu X_t dt + (x_0 - X_{t^-}) dN_t \]

where \( X_0 = x_0 \)

Fokker-Planck Equation:

\[ \frac{\partial f_X}{\partial t} = -\frac{\partial}{\partial x} (\mu x f_X) - \lambda f_X \]

Steady state CCDF – Pareto

\[ F_X(x) = \left( \frac{x}{x_0} \right)^{-\frac{\lambda}{\mu}}, \quad x \geq x_0 \]
Sample Path and Density (Pareto)

- Connection to Tan’s work
  - example: exponential packet lengths

\[ \text{sample path of } dX_t = \mu X_t dt + (x_0 - X_t) dN_t \]

\( \mu = 5 \)
\( \lambda = 3 \)
\( x_0 = 7 \)

\( \text{latency} \)

\( \text{pkt length} \)
Sample Path and Density (Pareto)

- Social network degree distribution
  - exponentially distributed user ages (social network ages)
Sample Path and Density (Pareto)

- Connection to Tan’s work
- Social network degree distribution

Sample path of $dX_t = \mu X_t dt + (X_0 - X_t) dN_t$

- $\mu = 5$
- $\lambda = 3$
- $x_0 = 1$

Pareto distribution density
Sub-exponential Growth with Exponential Restart

Sub-exponential growth, i.e. $\delta \in [0, 1)$

\[ dX_t = \mu X^\delta_t dt + (x_0 - X_{t^-})dN_t \]

Fokker-Planck Equation is

\[ \frac{\partial f_X}{\partial t} = -\frac{\partial}{\partial x} (\mu x^\delta f_X) - \lambda f_X \]

Steady state CCDF – Weibull (headless)

\[ \bar{F}_X(x) = \exp \left\{ -\frac{\lambda}{\alpha(1-\delta)} (x^{1-\delta} - x_0^{1-\delta}) \right\}, \quad x \geq x_0 \]
Sample Path and Density (Weibull)

Sub-linear Preferential Attachment

sample path of $dX_t = \mu X_t^{\delta} dt + (X_0 - X_t) dN_t$

$\mu = 5$
$\lambda = 3$
$X_0 = 1$
$\delta = 0.5$

density of Weibull distribution
Geometric Brownian Motion with Exponential Restart

Geometric Brownian motion with exponential restart

\[ dX_t = \mu X_t dt + \sigma X_t dW_t + (x_0 - X_{t-}) dN_t \]

where \( X_0 = x_0 \)

Density \( f_X(x, t) \) satisfies Fokker-Planck Equation

\[ \frac{\partial f_X}{\partial t} = -\frac{\partial}{\partial x}(\mu x f_X) + \frac{1}{2} \frac{\partial^2}{\partial x^2}(\sigma^2 x^2 f_X) - \lambda f_X \]
Steady State Density – Double Pareto

Steady state density – Double Pareto

\[ f_X(x) = \begin{cases} 
  x_0^{-1} \frac{\alpha \beta}{\alpha + \beta} \left( \frac{x}{x_0} \right)^{\beta - 1}, & x \in (0, x_0] \\
  x_0^{-1} \frac{\alpha \beta}{\alpha + \beta} \left( \frac{x}{x_0} \right)^{-\alpha - 1}, & x \in [x_0, \infty) 
\end{cases} \]

where \( \alpha, -\beta (\alpha, \beta > 0) \) are roots of quadratic equation

\[
\frac{1}{2} \sigma^2 \gamma^2 + \left( \mu - \frac{1}{2} \sigma^2 \right) \gamma - \lambda = 0
\]
Sample Path and Density (Double Pareto)

MySpace degree distribution

![Graph showing the distribution of MySpace degree distribution with a log-log scale on the axes. The x-axis represents the number of friends, ranging from 1 to 10^6, and the y-axis represents the complementary cumulative distribution function (CCDF), ranging from 10^-5 to 1. The data points form a power-law distribution.]
Comparison of Three Densities

densities of double Pareto, Pareto and Weibull distributions

\[ f(x) \]

\[ x \]

\[ 10^{-9} \]

\[ 10^{-8} \]

\[ 10^{-7} \]

\[ 10^{-6} \]

\[ 10^{-5} \]

\[ 10^{-4} \]

\[ 10^{-3} \]

\[ 10^{-2} \]

\[ 10^{-1} \]

\[ 10^{0} \]

\[ 10^{1} \]

\[ 10^{2} \]

\[ 10^{3} \]

\[ 10^{4} \]

\[ 10^{5} \]

\[ 10^{6} \]

\[ 10^{7} \]

\[ 10^{8} \]

\[ 10^{9} \]
Conclusions

- SDEs unified way to generate power law and non-power-law distributions
- Insight about when power laws occur
- SDEs flexible and moments of distributions easy to obtain
- Connections to
  - Tan’s work
  - Social network degree distribution